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Essays on Jacob Klein

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8 Editorial introduction

Jacob Klein and the *Mathesis Universalis*

Giovanna C. Cifoletti

"All teaching and all intellectual learning (*mathesis*) come about from already existing knowledge".[1] The beginning of *Posterior Analytics* was a *topos* in the sixteenth century. Aristotle synthesizes many things in these few words. Interpreted in early modern and modern terms, this statement contains the questions we are going to discuss in this section of the journal containing four articles. What matters for us is not so much what Aristotle meant by the sentence[2] in the epigraph, but how sixteenth-century mathematicians and mathematical practitioners understood this text. Jacob Klein, in his two essays,[3] developed a history of number and counting from the Pythagoreans to Wallis' *Mathesis Universalis* (1657). The transition from ancient to modern times is presented as François Viète's achievement, and at the end of the chapters devoted to this mathematician Klein draws a sketch of the role of *mathesis universalis* in the sixteenth century. The title of this chapter is "The reinterpretation of the *katholou pragmateia* as Mathesis Universalis in the sense of *ars analytice*". In order to investigate the foundations of Viète's *Logistica speciosa*, he cites the passages in Aristotle introducing a *mathesis* capable of a general treatment. The references are mostly to Aristotle's *Metaphysics*, where *mathesis* is included in first philosophy and therefore is able to give a justification of principles, including common notions. Klein also gives one reference to *Posterior Analytics*, which points in a different direction.

Reading Klein's GMTOA has been influential for many historians of mathematics, including myself. These pages shed a new light onto the philosophical role of symbolic algebra in the construction of modern science. Starting on this basis, I have developed a two-sided investigation of sixteenth-century mathematical and dialectical works, in particular those influenced by the rhetorical reform of logic and the linguistic turn promoted by Lorenzo Valla.[4] I found that both groups payed special attention to

1 Aristotle. *Posterior Analytics*. I, 1. Quotations are from the translation by Jonathan Barnes, printed in Oxford in 1975.

2 About Aristotle and the role of mathematics in the *Posterior Analytics*, see Orna Harari. *Knowledge and Demonstration. Aristotle's Posterior Analytics*. Berlin, 2004.

3 "Die Griechische Logistik und die Entstehung der Algebra" (1: Berlin, 1934, II: Berlin, 1936, from now on cited in the translation by Eva Brann, *Greek Mathematical Thought and the Origins of Algebra*. M.I.T. Press, 1968, hereinafter "GMTOA."

4 Giovanna Cifoletti, *The Art of Thinking Mathematically*, special issue of *Early Science and Medicine*, 11, 4, 2006, pp. 369–477, "Mathematics and Rhetoric. Introduction", pp. 369–389 and "From Valla to Viète: the Rhetorical reform of Logic and its Use in Early Modern Algebra", pp. 390–423.

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Posterior Analytics, as a context in which rhetorical or dialectical syllogisms are acknowledged as valid, and also as a source of references for sixteenth-century programs of mathematical sciences. These debates took place in the context of European liberal arts faculties and abacus schools. These two kinds of teaching institutions were not so separate anymore: by the end of the fifteenth century, the same person would indeed teach at both schools. As we shall see, this will involve a transfer of theoretical elements from mathematical practice to theory and from philosophy to mathematical teaching. Following Klein and my study of early modern mathematical and dialectical texts, there are four interpretations, not mutually exclusive, of the sentence in the epigraph.

1 The first interpretation takes previous knowledge as a reference to innate knowledge. This corresponds to Aristotle's treatment of principles in *Posterior Analytics*.[5] Aristotle discusses how to structure scientific discourse: principles are the necessary basis for building demonstrations and demonstrative science because without principles there would be an infinite regress of proofs. Aristotle's examples of demonstration are taken from mathematics. In fact, sixteenth-century mathematicians considered Aristotle as a significant mathematical author, and they were committed to an encyclopedia in which mathematics had a special role as a major example of *mathesis* or intellectual learning. The examples of principles given by Aristotle are Euclidean definitions, postulates and common notions or axioms and the theorems derived from them assumed as principles for other demonstrations. Common notions are principles common to different mathematical sciences. In this sense, they are an exception to the rule of the separation of sciences by genus, and they are exceptions to Aristotle's criticism of innate knowledge.

Because of their sensitivity to Neoplatonist and Stoic suggestions, sixteenth-century mathematicians appreciated this exception and stressed this kernel of innate knowledge admitted by Aristotle. Campanus' version of Euclid reinforced this interpretation. Many passages of *Posterior Analytics* concern the role of principles for all the mathematical sciences as well as the reciprocal articulation of mathematical sciences with "mixed" sciences. In 75b15, Aristotle asserts that

> One cannot prove by any science the theorems of a different one, except such that they are so related to one another that the one is under the other –e.g. optics to geometry and harmonics to arithmetic.

An important criterion of the subordination of sciences is whether we deduce the *fact* or the *reason why*:

> The *reason why* differs from the *fact* when each is considered by means of a different science. And such are those which are related to each other in such a

5 For a reading of *Posterior Analytics* in sixteenth-century dialectic, see Neal W. Gilbert. *Renaissance Concepts of Method*. New York, 1960; Charles B. Schmitt. *Aristotle and the Renaissance* (Cambridge, MA, 1983) and the recent Lodi Nauta. *In Defence of Common Sense: Lorenzo Valla's Humanist Critique of Renaissance Philosophy*. Cambridge, MA, 2009; Sandra Bihlmeier. *Ars et Methodus. Philipp Melanchthon's Renaissance Concept of Philosophy*. Göttingen, 2018, also in consideration of the role of mathematics in reformed education promoted by Melanchthon.

way that *the one is under the other*, e.g. optics to geometry, and mechanics to solid geometry, and harmonics to arithmetic, and star-gazing to astronomy. Some of these sciences bear almost the same name –e.g. mathematical and nautical astronomy, and mathematical and acoustical harmonics. For here it is for the empirical <scientist> to know the *fact* and for the mathematical <to know> the *reason why*. For the latter have the demonstrations of the explanations, and often they do not know the *fact*, just as those who consider the universal often do not know some of the particulars through lack of observation.

(78b35, the italics are mine)

Notice that subordination of mathematical sciences also induces different degrees of certitude:

One science is more certain than another and prior to it both if it is at the same time of the *fact* and of the *reason why* and not of the *fact* separately from the science of the *reason why*.

(87a32)

Certitude has to do with cause. Yet, there is another criterion of order among sciences, that is, the independent being of the objects.

And if <a science> is not said of an underlying subject and the other is said of un underlying subject (e.g. arithmetic and harmonics) and if it depends on fewer items and the other on an additional posit (e.g. arithmetic and geometry). (I mean as an additional posit, e.g. a unit in a positionless reality, and a point a reality having position – the latter depends on an additional posit).

(87a34)

Even only from these few passages, we see that Aristotle describes an encyclopedia of mathematical sciences. These passages also define, though indirectly, what kind of mathematics is at work in all the mathematical sciences, and what kind of mathematics is the sap of the tree of knowledge. This is what sixteenth-century mathematicians and mathematical practitioners were looking for.

Principles are hierarchical; arithmetic and geometry cannot interchange principles. However, there are some common principles for all the mathematical sciences, that is, common notions, and there are theorems transmitted from one science to its subordinate science:

Of the things they use in demonstrative sciences some are proper to each science and others common –but common by analogy, since things are useful insofar as they bear on the kind under the science. Proper: e.g. that a line is *such and such* and straight <so and so>; common: e.g. that if equals are taken from equals, the remainders are equal.

(76a38)

Here, we see an example of common notion made explicit also in Euclid's *Elements*. It has several implications because it introduces the idea of calculation (in this case, subtraction, but there is a similar common notion for addition) between quantities that can be arithmetical or geometrical, that is discrete or continuous.

This point authorized sixteenth-century mathematicians to develop under Aristotelian auspices the ancient idea of a *whole mathematics* or *mathesis universalis*, traceable to the authority of Iamblichus and Proclus. Proclus in particular made explicit the list of common notions and of the theorems which were their immediate consequences. They concerned arithmetic as well as geometry and in consequence their subordinate sciences.

2 This leads us to the second interpretation of Aristotle's previous knowledge, that is, the existence of a pre-mathematical doctrine prior to arithmetic and geometry but partially translatable into both. This interpretation was popular among the algebraists. Pre-existent knowledge can also be a whole *general* theory existing as a precondition of specifics theories. This theory would be constituted of the most general axioms and a few more statements depending on them. We can associate it, as the mathematician Niccolò Tartaglia did in 1557, with the first passage of chapter 10 of *Posterior Analytics*:

> I call principle in each kind those which it is not possible to prove to be. Now both what the primitives and what the things dependent on them signify is assumed; but that they are must be assumed for the principles and proved for the rest -e.g. we must assume what a unit or what straight and triangle signify, and that the unit and magnitude are, but we must prove that the others are.

While no science of principles is possible, it is possible to build a science on a minimal extension of principles. This is the case in this interpretation: that is, to develop Aristotle's and Euclid's common principles into a primitive discipline, *mathesis*, based on innate faculty of counting and therefore prior to arithmetic and geometry. This is theoretical logistics, that is, computation with new numbers that are not understood to be composed of units, but are rather seen as general quantities susceptible to calculation. Any kind of number is admitted, also irrational numbers, and this science is about a general number, a number not definable in terms of its units. Fibonacci mentions it in the *Liber Abaci*, associates it with Book II and identifies it with algebra. *Algorismus* (i.e. the genre of books teaching Arabic numerals and their computations) and algebra introduced in the Latin West a new concept of number that will take definite shape in sixteenth-century algebraic symbols. In fact, the mathematics of Arabic digits coming from the Arabic language included a philosophy. This philosophy posited the foundations of algebra on the basis of the "innate" faculty of counting and the common notions. It also involved a new way of seeing empirical sciences, because a general number can represent any quantity, any number or magnitude, or entity like sound or ray of light or motion of a star.

3 This brings us toward the third interpretation of Aristotle's appeal to previous knowledge and it is another way in which sixteenth-century algebraists looked at their discipline: they saw algebra as a mathematical discipline of non-mathematical, physical objects. Al Farabi described this science as *De ingeniis*, in which **number is connected with matter.** *Posterior Analytics* and *De Caelo* could justify this view: *De ingeniis* was a good example of a discipline in the Aristotelian tree of knowledge: a mathematical discipline which makes use of common principles but does not use some specific principles of other sciences; it does not mix genera. Notice also that Aristotle's statement about previous knowledge uses the term

mathesis (which can be translated as intellectual learning), which has the same etymology as mathematics. So the passage talks about any kind of knowledge as intellectual learning. However, for sixteenth-century scientists, who had studied Greek in order to read Aristotle and Euclid, *mathesis* meant mostly mathematics but a mathematics that had gained the most fundamental role in knowledge and science; *mathesis* pointed to the fact that there is a kernel of *mathesis* whose development constitutes mathematics, so that any knowledge is at its basis mathematical. Sixteenth-century authors could legitimately connect the word *mathesis* with the mathematical references used by Aristotle and interpret the passage in the light of *prima mathesis* or general mathematics. This had been mentioned by Iamblichus in the Pythagorean tradition and will be called later *mathesis universalis* also by Descartes and Leibniz. In this context, the *prima mathesis* would provide a connection between the pre-existing knowledge and new knowledge. Already Giovanni Crapulli[6] made the point that it is necessary to distinguish between many similar expressions: *mathesis* is used commonly and it often means mathematics; *mathematica universa*, which means the whole of mathematical disciplines, as well as *mathesis universalis*, *scientia mathematica communis*, *mathematica generalis*, finally *prima mathesis*. David Rabouin[7] has refined the history of this tradition further following its development from the rediscovery of Proclus' *Commentary of the first book of the Elements* in the sixteenth century to Descartes and Leibniz. After Kant, this mathematical project took on a new life with Husserl, who described *mathesis universalis* as science of the object in general, and originated in early modern symbolic algebra. Ernst Cassirer emphasized the role of symbolic algebra in another direction, highlighting the concept of function.

This is the modern tradition of *mathesis universalis*, and here we come to the core of *mathesis universalis* for Jacob Klein. The revolution with respect to antiquity is general quantity eventually interpreted as general number or general magnitude and expressed by a sign. This sign will behave as a number in calculation. According to Klein, this combination of the constitution of a general concept and of an object of calculation is due to the articulation between first and second intentions, an important Medieval distinction.

Furthermore, on Klein's view, the *prima mathesis* would be the development of pre-existing knowledge into the new knowledge deduced from the former. In this sense, *mathesis* is called *universalis* not only because it contains some basic principles that a few sciences have in common, but because it can solve all problems and its range of application is unlimited. This is what is implied by René Descartes' *Regulae ad directionem ingenii*, the work he wrote in an ideal dialogue with Aristotle and his *Posterior Analytics*.[8] I have claimed elsewhere that the *Regulae* were the result of Descartes' understanding of the French algebraic tradition and introduced his program by proposing to translate all problems into equations, whereas the rules were intended to teach how to perform this

6 Giovanni Crapulli. *Mathesis universalis. Genesi di un'idea nel XVI secolo*. Rome, 1969.

7 David Rabouin. *Mathesis Universalis. L'idée de mathématique universelle d'Aristote à Descartes*. Paris, 2009.

8 Jean-Luc Marion stated this thesis in his *Sur l'ontologie grise de Descartes*. Paris 1993.

translation.[9] Recent historiography has provided new information about the Medieval mathematical contributions to this grand program as well as about some philosophical suggestions based on algebra.

4 The fourth interpretation of "previous knowledge" points to what I take as the strong thesis of Klein's work: that symbolic algebra could only happen as a result of the historical apprehension of ancient mathematics newly rediscovered. By means of their consciousness of past mathematics, sixteenth-century practitioners would have been in the condition of reasoning with general numbers as concepts of concepts. With Viète's letters for unknowns and known quantities, the concept of number has lost its connection with a multitude of beings and has become an indeterminate quantity, but capable of being counted: that is, its meaning depends on its connection to other concepts through operations. This is the step allowing Viète to give these new numbers a form, and to attribute them a sign. The sign of this concept of concepts, or number of numbers was not a digit but, going back to the previous knowledge, it was a letter. The combination and computation of numbers and letters, of known and unknown quantities, constituted a formula.

According to Klein, the peculiar relation between sixteenth-century algebraists and ancient mathematical texts would be at the origin of the shift in conceptuality leading to the novel concept of general number. They presented their mathematics as new but at the same time old, because they felt they had reconstructed an intellectual technique which had been Greek. Many of them actually claimed that what they did was the ancient doctrine which had been consciously hidden by the Greeks. According to Klein, Renaissance algebraists did not make explicit the shift they introduced because they were not fully aware of it. This could have been the cause of the problem of meaning in symbolic algebra and more recently in the formalism for physical sciences using that language.

Klein was a philosopher of science, a mathematician and a classics scholar. When he published his two essays later translated into GMTOA[10] about the origins of algebra, he declared from the start that its purpose was not a clarification of this part of mathematics in and for itself. Rather, he planned to look for "the sources, today almost completely hidden from view, of our modern symbolic mathematics" (GMTOA, p. 4) and to investigate the origin and conceptual structure of its formal language because

> The intimate connection of the formal mathematical language with the content of mathematical physics stems from the special kind of conceptualization which is concomitant of modern science.
>
> (ibid.)

Klein published his essays in Otto Neugebauer's journal. Neugebauer not only was committed to the discovery of Mesopotamian mathematics, but specifically

9 Giovanna Cifoletti. "Quaestio sive Aequatio. The Notion of Problem in Descartes' Regulae", in *Mathematics and Rhetoric. Jacques Peletier, Guillaume Gosselin and the Making of the French Algebraic Tradition*. Princeton U. Ph.D. Ann Arbor, UMI, 1992, www.proquest.com.

10 Otto Neugebauer and the editorial board of the journal *Quellen un Studien in Geschichte der Mathematik, Astronomie und Physik* had moved to Copenhagen: on the role of Otto Neugebauer, see A. Jones, C. Proust, J. A. Steele. *A Mathematician's Journey: Otto Neugebauer and Modern Transformations of Ancient Science*. Zürich, 2016.

to its inclusion in the history of algebra. Algebra was at the time at the center of mathematical and philosophical research, and also of historical research. In spite of the fact that the very name of algebra had the connotation of its Arabic origin, a few authors, including Neugebauer, maintained that algebraic thought originated together with writing and its development in various civilizations, and that it can be studied as entirely abstracted from them. Klein instead made clear the radical discontinuity between the Greek notion of number and what replaced it in early modern times. This radical discontinuity needs to be brought to light in order to uncover what is presupposed by modern symbolic logic, mathematics and mathematical physics.

We can take Klein's thesis on early modern algebra as a starting point for a different perspective on the history of mathematics, by taking seriously its philosophical context.

According to Husserl's late philosophy, the historicity of the meaning of the basic concepts of the exact sciences is crucial for transcendental phenomenology's program of providing philosophical foundations for the exact sciences. Burt Hopkins has shown that Klein has gone further than Husserl on two main points. First, with respect to *Crisis*, because he recognized that a science cannot be justified by itself, but only from outside; second, because he produced the first historical phenomenological study to *desediment* the origin of the concept of number.

After the publication of Hopkins' groundbreaking work on Klein, *The Origin of the Logic of Symbolic Mathematics:Edmund Husserl and Jacob Klein*, I discussed it in my teaching. I invited its author to the *Ecole des Hautes Etudes en Sciences Sociales* and to my seminar. We present here the outcome of our collaboration and public workshops from 2014 to 2018. Gabriele Baratelli joined us in 2018.

We articulate this new perspective into four investigations:

1 Gabriele Baratelli gives a survey of the main philosophical stakes involved in the phenomenological study of *mathesis* and algebra: clarification of concepts requires to trace their constitution. He completes the picture with a reference to the past, Kant and neo-Kantian themes on the one hand and more recent philosophical challenges on the other hand. Baratelli explains in detail Klein's contribution to the clarification of the conceptuality of number in ancient and modern times, stressing Klein's role in interpreting, criticizing and correcting Husserl's perspective.
2 Burt Hopkins explains why Klein considered his work as representing a radically new approach to the philosophy and history of mathematics, with a focus on his philosophical account of the historicity of number, overcoming the assumption of mathematical concepts as a-historical invariants. The motivation for this study was Klein's conviction that there is an intrinsic connection between formal mathematical language and the content of mathematical physics. Hopkins unpacks Klein's historical-philosophical argument that there can be no foundation of an exact science within the science itself. Klein's study of number starts from a disengagement from the contemporary mode of thought and leads him to general number and indeterminate quantity at the origin of Viète's algebra.
3 I explore the philosophy of algebra appearing in the algebraic texts by Luca Pacioli and Niccolò Tartaglia and their legacy. I apply Klein's method of investigation

and take into consideration these two mathematicians: this reinforces Klein's theses on the shift in conceptuality of early modern algebra. I suggest a wider Medieval and Renaissance perspective to view the contributions of these mathematicians, and thus situate it within the debates on dialectic in general and on *Posterior Analytics* in particular. Moreover, I articulate some of the interpretations above in dialogue with Klein's findings.

4 Jean-Marie Coquard examines what Klein has said and not said about Simon Stevin. Klein prepared the way for the comparison between Stevin's and Viète's algebra. In particular, Coquard studies the role of history in Stevin's thought about science and in Klein's research. Stevin's concern with history allowed him to recognize the role of Arabic mathematics with respect to the position numeral and decimal system. Stevin is also one of the main early modern authors reconfiguring the function of theory and practice, contemplation and matter in the mathematical sciences. Klein did not study Stevin's *mathesis* program but Coquard's discussion of it belongs to Klein's legacy.

These papers evoke the challenges of the times in which they are produced, as did the actors in these historical investigations. At the time of François Viète and Simon Stevin, symbolic algebra could be used for astronomy, as Viète did, but also for engineering as Stevin, or for music and optics, as Descartes. Jacob Klein in the first third of the twentieth century was facing the challenges of the foundations of mathematics debated by logicians and mathematicians on the one hand and the foundations of mathematical physics debated by physicists and philosophers, including Husserl, on the other hand. Since then, deterministic science has declined, probability and statistics have become mainstream and the use of symbolic mathematical language appeared in models, from physics to economics. Now the challenge is to investigate philosophically the foundations of artificial intelligence and its algorithms. The program of *desedimenting* the sense of formulas is all the more pressing, as it is crucial to understand them in all their implications.

9 How letters become symbols

Jacob Klein's genealogy of formalization

Gabriele Baratelli

Abstract: The chapter aims to give a critical presentation of Jacob Klein's study on the origin of the modern process of formalization. It is divided into three strictly related parts. In the first one, I set forth Klein's general motivation and philosophical principles that guide his investigations in the history of mathematics. I argue that entire perspective can be consistently interpreted as "critical" in the sense I introduce. In the second part, I take into account Klein's specific thesis concerning the "exemplary case" that he considers, namely the difference between the Greek and the modern formalized concept of number. In the last part, I draw some conclusions from the two first moments of my chapter in order to offer a plausible interpretation of Klein's account of the transformation of the concept number and to propose some brief explanatory philosophical consequences.
Key Words: Jacob Klein; Formalization; Concept of Number; Symbolic Language; Historicity of Formal Sciences; Modern Conceptuality.

9.1 Introduction

Jacob Klein wrote two substantial articles published in 1934 and 1936 entitled *Die griechische Logistik und die Entstehung der Algebra*, and then translated in 1968 as *Greek Mathematical Thought and the Origin of Algebra* (hereafter *Math Book* or *MB*). In both German and English, the book's esoteric title is not only dry but also dissimulative of its actual subject. It suggests that the content of the book is a scholarly and exclusively historical examination of an episode in the history of mathematics. To be sure, it is partly this, as the reader discovers in the book's labyrinth of quotations and long series of philological analyses. The historical considerations seem always to be in the foreground and are indeed easier to grasp than its submerged themes. Today, historians generally grant his work the enduring merit of having discovered the real nature of numbers in Greek thought, 40 years before a wider rediscovery was initiated by Sabetai Unguru's 1975 "*On the Need to Rewrite the History of Greek Mathematics*." As we shall see, however, Klein's achievement could only be realized, thanks to his employment of specifically *philosophical* conceptions of *history* and *language*. The philosophical nerve of Klein's work, albeit often concealed, constitutes its internal energy and the motive that guides the entire investigation, such that a failure to take this into account cannot but result in an insufficient understanding of its singular accomplishment.

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In "Phenomenology and the History of Science," a paper written in 1940 in honor of Edmund Husserl,[1] Klein acknowledged the "historical empathy" that guides the latter phase of Husserl's thought's engagement with the constitution of modern science. Specifically, Klein agrees with Husserl that "the problem of the origin of mathematical physics is the crucial problem of modern history and modern thought" (*LE*, 79). Moreover, since "the establishment of modern physics is founded upon a radical reinterpretation of ancient mathematics" (79), the history that Klein relates regards the assimilation and the simultaneous transformation of the eidetic objects of Greek *episteme* into the symbolic character of the algebraic language that drives the mathematics behind the modern project of a *mathesis universalis*. On Klein's view, then, "this universal science bears from the outset a *symbolic* character" (81): it is this "character" and the "logic" that governed the original functioning of the new powerful scientific language, which is the focus of Klein's philosophical and historical interests.

At the time Klein wrote his *Math Book*, the question concerning the formal nature of modern science was understandably focused on *physics* and on its formidable power of explaining, predicting and controlling natural events.[2] After more or less 85 years, these concerns seem to have faded away. The "crisis" felt by many is not at the center of the debate anymore. Analogously, the motives that encouraged the investigation of such arcane disciplines, like the history of algebra in Klein's case, seem to have lost their critical significance as well. However, despite this superficial lack of interest, we argue that the reasons why the question of "symbolism" was pervasive at the time among philosophers working in various traditions remain valid today. They are especially visible nowadays once we put the social sciences under the spotlight, especially *economics*. Of course, the role that it plays today can hardly be overestimated, just like the fact that its use as a theoretical instrument to explain, predict and control social events is clearly grounded in its institution as a (at least apparent) mathematical, i.e. *symbolic*, science.[3] Even though it does not reach the status of scientific universality that physics does,[4] it aspires nonetheless to gain an "objective" status in virtue of the formal exactness given by mathematical formulae. Thus, *mutatis mutandis*, the problem of the "sense of the formulae" still holds, even if with different gradations. Economical doctrines are instruments of power whose legitimacy also relies upon their "scientificity." They determine political decisions according to an oracular authority that pertains to those who can manage the technical language and the methods of science. As Simone Weil noted, this state of affairs is analogous to that of archaic societies, when knowledge was precluded to the many and secured

1 The paper is now in the book *Lectures and Essays* (hereafter: *LE*).

2 In 1935, thus contemporary to Klein's and Husserl's writings, even Heidegger gave a course (now titled *The Question Concerning the Thing*) where he took into account the very same problem by studying the conceptual premises embedded in Newton's laws.

3 These themes are brilliantly developed by Düppe (2011).

4 This is not only due to its matter of investigation, less easily prone to quantification and experiments, but also due to the peculiar ideological nature of economics. As Marx said in the preface of the first edition of the *Kapital*:

> In the domain of Political Economy, free scientific inquiry meets not merely the same enemies as in all other domains. The peculiar nature of the materials it deals with, summons as foes into the field of battle the most violent, mean and malignant passions of the human breast, the Furies of private interest.

by the priests who had a privileged relationship with the Gods.[5] Not only is this true today – with the obvious difference that there are no Gods, and that scientists are now the vectors of the mundane objectivity carried by formulae – but it is even legitimate to suppose that scientists themselves, like priests, do not have an authentic cognition of their own doctrines. Although their lack of awareness is dissimilar to the complete ignorance of the so-called "people," they are subjected to the peculiar oblivion inevitably caused by symbolism, whereby, as Husserl has taught us, their knowledge easily degrades to a mere theoretical technique. Since economics has direct material effects through the way in which it functionally organizes our lives, the critical exposition of its buried ontological assumptions and the origin and limits of its objectivity is a necessary duty. At the end, Klein's historical-critical examination has an ethical, political and even pedagogical import. It aims to dismiss the "symbolic unreality" within which we are immersed, and so to contest its objectifying force.[6]

Klein says: "There may be many ways to overcome this symbolic unreality. One of these ways is to understand how ancient science approached the world" (*LE*, 64). Greek *episteme* is, in Klein's hands, the *other* of modernity, its mirror. It represents the original locus where scientific discourse was preserved in natural language and formulae were not conceived. Modernity, instead, wrested away letters from words and uncannily employed them as mathematical symbols. In this way, the problem articulated by Klein, i.e. the origin of symbolic thinking, can be understood as the last episode of the complex history of alphabetic characters. Derived from ancient logograms, letters are finally deprived of any meaningfully sensible illustration in the Greek alphabets. The letter "A" does not depict the head of a bull anymore:[7] its body is completely irrelevant for the transmission of the meaning and it is reduced to a formal typographical utility. Their sense is bound to the relationship to the voice and on their way to efficiently analyze it through consonants and vowels. Letters, like linguistic atoms, succeed in abstractedly dividing the original unity of the syllable.[8] This original usage has been transformed once they have been used algebraically.

> What characterizes words is the union of their sound with their functioning as signs. This union can be broken: both sound and sign may become autonomous.

5 I am indebted to Ginevra Scarcia for having recommended to me this and the other remarkable Weil's passages.

6 In Klein's words:

> the features of the *mathesis universalis*, which appear most forcefully in our Science of Nature and dominate our entire manner of thinking can, I trust, be traced in the social and economic fields in which we live. Along the lines of our society, every one of us must 'do his job' according to certain rules imposed on us by ever-working machineries. The production and consumption of goods have required a sort of 'automatic' character. No one can escape the fatality which is the result of this automation. Our life, then, even our most intimate life, is completely conditioned by social and economic necessities which are alien to ourselves and which nevertheless accept as the true expression of ourselves. Our work, our pleasures, even our love and hatred are dominated by these all-pervading forces which are beyond our control.
>
> (*LE*, 63–64)

7 See Herrenschmidt (2007: 33–40).

8 Sini (1992, 1994) argues that the practice of definition typical of the rational discourse reflects explicitly what is already implicitly done in the practice of writing and "thinking" alphabetically, that is, in the practice of atomizing the syllable.

> The naked signs, turns later on into symbols, constitute the skeletal language of mathematics.
>
> (265)

This is, in a nutshell, the question of formalization. How is this detachment possible? How can letters leave their natural role as elements of words and gain a mathematical independence? Thanks to this operation, we can now recognize, without a second thought, as valid the addition "$n + n = 2n$." But what are we doing? Do we add letters? Or numbers? Or numbers of a special kind? Once we consider this operation without presupposing its legitimacy, it appears strange and even absurd.

Two paths can be followed at this point.

> At one extreme, formalization is understood as the employment of letter signs or other marks to, at the very least, "stand for" or "symbolize" any arbitrary object or content – "whatever" – belonging to a certain "domain". […]. At the other extreme is the view that formalization is the fulcrum for an unprecedented transformation in how the science of the so-called West forms its concepts, a transformation that is as all-encompassing as it is invisible to this day.
>
> (Hopkins, 2011: 3)

Klein takes the second one. In this work, we try to offer a general outlook of the reasons why Klein chose this view and of the philosophical arguments through which he justified it. We have divided the paper into three closely related strata. In the first one, we show the Kleinean theoretical tools that make *visible* the problem of formalization. They overcome the enchantment caused by the symbolic techniques and open the possibility to critically examine it. The second part looks more closely at Klein's exemplary case of the *Math Book* and presents it *statically*. We take into consideration the abstract differences between the concepts of number in Greek thought and modernity without focusing on their mutual relationships. After a *pars destruens* where we critically discuss what we call the "anachronistic accounts" that wrongly conceptualize the history of mathematics, we finally interpret Klein's own account of the symbolic transformation of the concept of number in modernity based on the reading of Vieta's mathematical work. This last part is intended to present the *dynamic* version of the theme set in the second one: employing the theoretical instruments introduced in the first part, we focus on the structure of the passage, showing how the Greek concept of number was algebraically transformed and explaining why this very transformation passed unnoticed.

9.2 Klein's philosophical account as a critical assessment

9.2.1 What is a critique?

Jacob Klein's study of the origin of modern conceptuality can be read consistently through a Kantian lens. Specifically, it has been argued (Hopkins, 2014) that the notion of "critique" is a useful key to organize the complex net of historical and philosophical ideas contained in the *Math Book*. In general, a "critique" is an "assessment of a cognitive faculty or capacity that does not 'aim at the extension of cognition itself' but aims at its 'purification' in a manner that would keep it 'free of error'" (364).

This is the case because a critical project does not focus on the *content* of a certain "dogmatic" doctrine, but rather on the *way* in which it is made possible.

> The dogmatic procedure in general is characterized by the attention to the content, to what is said, and puts forward its arguments by opposing one content to another [...]; the critical procedure instead focuses on the *mode* in which a determinate affirmation is conducted: the proposition is left to its value or lack of value, and does not claim to know the object better, but interrogates its "conditions of possibility."
>
> (Chiurazzi, 2017a: 75)

Thus, it does not add new information on the level of the "what" (e.g. it does not say, in the case of mathematics, whether a certain conjecture is a theorem or not), but it does show the unseen "how" that renders a certain "what" a "fact" (of a mathematical kind or otherwise).

9.2.2 The **Begrifflichkeit** *as Klein's unseen how*

We argue that Klein's project instantiates this general critical framework. In his case, the specific characterization of the "how" that is to be investigated takes the form of the notion of *Begrifflichkeit*. Although this concept arguably sets the plan of his theorical undertaking and represents the very end of it, Klein does not make it satisfactorily explicit. In "The World of Physics and the 'Natural' World," a lecture delivered in 1932, it is vaguely defined as "*the way* in which these contents are to be interpreted, *the way* [my emphasis] in which the concepts *intend* what is meant by them whenever they are employed" (*LE*, 6). The fact that this pivotal notion has not been sufficiently clarified is reflected in the different choices of translation of the term.[9] Hopkins says that Klein "uses this term to refer to the characteristics that determine the basic concepts employed in mathematical and philosophical cognition" (2014: 368). Both definitions, despite their vagueness, enable us to link it to the Kantian notion of critique. In fact, they express the idea that what is at stake is not the objective nature of certain objects (e.g. the "essence" of numbers), but what at first allows their objectivity. Even though this "how" is surely connected to historical and psychological facts, it cannot be reduced to them. It is "a universal medium beyond reflection, in which the development of the scientific world takes place"[10] (*MB*, 121). As such, it permits certain mathematical or philosophical objects to appear in their peculiar mode of givenness.

To make this point clear, we can distinguish between *conception* and *conceptuality*. Examples of modern conceptions of numbers would be Vieta's or Frege's definitions

9 Eva Brann, the first translator of the articles written in German, chose the term "intentionality," whereas Burt Hopkins chose "conceptuality," like Iacopo Chiaravalli, the Italian translator ("concettualità"). Since Klein does not use the term *Intentionalität*, the alleged connection to the Husserlian doctrine that the word "intentionality" suggests is to be rejected. More likely, as Chiaravalli notes (2018: 16-17), the concept can be traced down to a Heideggerian source: Klein attended the lectures on Aristotelian philosophy in 1924, where his teacher extensively employed this concept. For a discussion on this question, see Hopkins (2011: 80, n7).

10 "*sie ist das allgemeine, von der Reflexion nicht mehr erreichte Medium, in dem sich der Aufbau der wissenschaftlichen Welt vollzieht*" (1936: 125).

of number. Having the status of psychological beliefs or "logical" theories, they can contradict each other and fight dogmatically for the conquest of truth. They try to say something about the nature of a certain object. However, Klein claims, they all have in common *ab origine* a conceptuality which lies beneath them as their condition of possibility. That is, a way of intending and encountering the thing originally, a pre-comprehension which guides and limits the multiplicity of historical conceptions and doctrines. Only because this pre-comprehension is presupposed and shared in by the same mode of formation of concepts, can they have a common ground suitable for the dogmatic debate.

9.2.3 Klein's main thesis

Klein's aim is to disclose the "understanding of the *particular Begrifflichkeit*, the *particular* character of the concepts with whose aid the mathematical physics which arose in the seventeenth century erected the new and immense theoretical structure of human experience over the next two centuries" (*LE*, 7). Some corollaries derive directly from this sentence. First of all, the definition implies the possibility of the existence of *multiple* conceptualities besides the "modern" one. Secondly, its "peculiar character" is strictly dependent on the constitution of mathematical physics, that is, on a historical fact. Therefore, to understand its structure, one needs to look at what historically happened in the sixteenth century, when "this new self-definition of the mathematical enterprise rest[ed] upon a certain re-interpretation of ancient mathematics (as of the whole of ancient science)" (*MB*, 117). The conscious recovery of ancient texts was for modern mathematicians not only a polemical weapon against the "empty" Scholastic science but also an authentic inspiring source. However, for the understanding of their conceptuality, this aspect is only one side of the coin. Their recovery was not "innocent" but involved "a characteristic modification of every ancient concept" (*LE*, 8). Since this modification is not recorded *apertis verbis* in a text like a conception is, it cannot be disclosed by a mere historical investigation. For example, Klein is not exclusively interested in what Descartes *actually* thought of his own mathematical project and of his philosophical debt with the Greeks. Nonetheless, even Descartes *shows* through his conceptions a peculiar conceptuality that is to be exposed otherwise. To underline the non-empirical side of the matter and to characterize the fact that it concerns the way in which the constitution of concepts is made possible, we will call this kind of transformation "transcendental." In the end, the structure of the origin of modern conceptuality is twofold: on the one hand, it consists in the *conscious historical assimilation* of Greek (mathematical) thought and on the other, in the *unconscious*, and in this sense forgotten from the beginning, *transcendental transformation* which accompanies it.

The *Math Book* is supposed to show the nature of this hidden transformation. The disclosure of the structure of Greek conceptuality is to be a part of the picture: Klein wants to make visible – at once – the *abstract difference* between it and the modern one, and the *dynamic relationship* between them. These two moments are mutually correlated and they will be developed in the second and third parts, respectively. Because the difference in question is not only a principle of distinction but even the condition of possibility of the transformation from one conceptuality to the other, it can be defined (for the reasons we shall explore below) as *incommensurable*. This dynamic connection is shown by analyzing the "exemplary" case of the concept of number in its mode of givenness in ancient Greece and its transcendental transformation in early modern mathematics and philosophy.

9.2.4 The motivation of the critical turn

Like in Kant's philosophy, the original stimulus for the investigation starts off with the highlighting of some irreducible contemporary dogmatic oppositions. The actual situation of knowledge, despite the prosperity of formal and natural sciences, is described by Klein as a conflicting one, fostering conceptual confusions and thus, alongside with them, "insurmountable difficulties" on the philosophical level. These phenomena are viewed, as it were, as the visible symptoms of a neglected disease. Now, symptoms can be "dogmatic," that is, they can include conflicts between "metaphysical" doctrines, like the one concerning the foundations of logic and ontology.

> The battle between the proponents of Peripatetic syllogistic and mathematical analysis about the primacy of their respective views concerning the framework of the world, which is thus initiated in the seventeenth century, is still being waged today, now under the guise of the conflict between "formal logic" and "mathematical logic" or "logistic," although its ontological presuppositions have been completely obscured.
>
> (*MB*, 169)

Klein's strategy does not consist in evaluating the respective arguments and "taking sides in the controversy." He does not want to *solve* a certain philosophical problem, but rather descending to its "origin," he aims at reveling its superficial and unsolvable nature. In other words, he *dissolves* the dogmatic contraposition, insofar as the study sheds light on the ontological roots responsible for it. While *ex ante*, that is, on the conceptual level of the extant situation of science, the opposition appears as unamendable, *ex post*, that is, after having exposed its ontological ground, it is recognized as going in circles: it is shown *why* the conflict is structural.[11]

Klein identifies two modern epistemological sources that determine the emergence of such dogmatical symptoms: the ones connected to language and history. On the one hand, the symbolic language of algebra that has become irreplaceable for exact sciences is ingenuously accepted as the natural means for scientific thought in virtue of its "exactness" – a property precluded to ordinary language. On the other hand, the historical development of science is retrospectively judged starting from the ingenuously accepted results of contemporary sciences, so that one cannot but find in the past mistakes or occasional discoveries finally systematized in an exact manner.

We shall see in detail Klein's critical divestment of these two concepts below. In both cases, though, they share the risk of embodying the overall "danger" brought about by the formalistic methods of modern science, i.e. "to confuse the symbolic means of our understanding with reality itself" (*LE*, 137). The two conceptions just mentioned, in fact, bring with them the *occlusion of a difference*. They produce a

11 "From a *critical* point of view, then, the *Math Book* is the genealogical account and critique – or better: the critique through a genealogical account – of the foundation of the conceptual impasses of modern science" (Majolino, 2012: 194).

reductio ad unum on a certain domain of reality, a reduction whose paradigm is to be found in mathematical physics. According to Klein, from the standpoint of modern science, we cannot but be confused into thinking that the *symbolic* concepts with which it operates disclose the real world behind the "apparent" one within which we live. As Husserl claimed in the *Krisis*, "we take for *true being* what is actually a *method*" (§ 9h).[12] The "world of physics" is thus at first enigmatically juxtaposed to the "natural" one, and eventually substituted for it. Klein holds that the shrinking of their difference is due to the symbolic nerve of modern conceptuality, so that the "danger" brought about by mathematical sciences is intrinsic to their language. Klein's first step of his critical project, then, is supposed to be the suspension of the granted epistemological validity of symbolic language. Analogously, the historical turn that he takes cannot assume as a matter of fact the modern concept of history, insofar as it is conceived as "an interpretation of historical 'movement' as such." It is the case because Klein thinks that this very concept, even when it is at the basis of the philosophical position of "historicism" or better "psychologism,"[13] was originally intended to "fill the gap between the ever more 'formalized' scientific approach to the surrounding world and our daily life" (*LE*, 72). In this respect, it is "the twin brother of mathematical physics" (71). As a result, it is only apparently a proper foundation or an actual opposition to symbolic science. History in this sense in fact *presupposes* modern conceptuality. But it is patent that any investigation that "starts out from a conceptual level which is, from the very beginning, and precisely with respect to the fundamental concepts, determined by modern modes of thought" *cannot* reach any authentic understanding of the origin of modern conceptuality itself (*MB*, 5).

In conclusion, once it is hypothesized that the reason of the internal dogmatic controversies stemming from modern science is to be found in its symbolic nature, the inquiry is then *critically* directed both to the logic of mathematical-formal language and to its historical roots.

9.2.5 The strategy or the question of language

The "logic of symbolic language," then, promises to give us the access key to the origin of modern thought. But what does "language" mean here and how can Klein justify his theoretical focus on such an exterior element? The second question is the easy one. Klein interprets the importance of symbolic language for modern science as follows: "The symbolic language of algebra, that is, the language proper to mathematical physics, is not a purely technical or instrumental matter. [...]. The mathematical method of our physics is inseparable from the very nature of this science" (*LE*, 61). This principle is the real cornerstone of his arguments. It avoids the ingenuous

12 "Mathematics and mathematical science, as a garb of ideas, or the garb of symbols of the symbolic mathematical theories, encompasses everything which, for the scientists and the educated generally, *represents* the life-world, *dresses it up* as 'objectively actual and true' nature" (*ibid.*). This profound philosophical commonality between Husserl and Klein is at the basis of Cosgrove's interpretation of the *Math Book* (2008a). A much more detailed study of the relationship between the two is in Hopkins (2011).

13 Klein maintains that historicism "is an extension and amplification of psychologism" since "any 'naturalistic' psychological explanation of human knowledge will inevitably be the history of human development with all its contingencies" (*LE*, 65-66).

conception that, in the end, language is a matter of *notation*, as if the symbolic transformation was merely occasioned by the needs of science to express more exactly a certain very abstract conceptual content. On this view, every notation is thought of as equally distant from the content, like different suits for the same body. Of course, an improvement in the precision of the linguistic means can be accomplished and it is even desirable. However, this is possible only when a certain language is already established and mastered – not when it is invented *ex novo*. Understanding its creation requires a different perspective. Being a new *language*, and not a more exact notation, the symbolic means of expression brings along *ipso facto* a new *conceptuality*. Since the former represents its visible part, the *Math Book* can be considered as a phenomenological exposition of the hidden conceptuality of symbolic mathematics through the critical interpretation of its linguistic body.

The critical attitude is necessary since the actual difference between symbolic and ordinary languages is occluded and then misunderstood. Presupposing the "*Faktum* of symbolic mathematics" (*MB*, 4) leads to the conception of natural language as an inefficient devise for expressing a putatively timeless mathematical-logical content. An easy way to point at the deeper distance can be directly derived from Klein's guiding hypothesis. That is to say, given that "the intimate connection of the formal mathematical language with the content (*Sinn*) of mathematical physics stems from the special kind of conceptualization (*besonderen Art der Begriffiichkeit*) which is concomitant of modern science" (4), a content expressed in that language cannot be easily transferred in the ordinary one. In truth, an "exact" translation is actually impossible: "It is a common mistake to believe that we can translate the theorems of mathematical physics into ordinary language, as if the mathematical apparatus used by the physicists were only a tool employed in expressing their theorems more easily" (*LE*, 61). According to Klein, the same goes even for simple sentences like:

1 Two units and two units make four units.
2 $2 + 2 = 4$

One can read the *Math Book* as the attempt to demonstrate that sentences like these are not equivalent. In the passage from (1) to (2), which seems at first glance innocent, a modification is concealed, a conceptual shift which is supposed, and which Klein's analyses endeavor to expose. It is hidden because we accept in advance as a matter of fact the perspective embedded in (2) without reading (1) in its own original and proper terms. If this is true for arithmetical sentences, it will be so *a fortiori* for fully formalized algebraic equations: "[algebraic formations] are comprehensible only within the language of *symbolic formalism*" (*MB*, 175). Hence, they cannot be translated exactly and properly understood in ordinary language.

This gap, however, is not an accident, a mere fact which depends only on the factual power of a certain language (alphabetical or formal) to express a certain content. Following Klein's main hypothesis, we can interpret the distance between the two languages otherwise. In his eyes, it is rather a "linguistic" symptom, alongside the dogmatic ones, that indicates a difference which can be now specified as "incommensurable." According to Chiurazzi (2013), the failure of an "exact" translation between two languages, i.e. a perfect match between "abstract" meanings, is a structural condition that recalls the one of mathematical incommensurability. In translation, in fact,

> certain meanings of a given language emerge as "inexpressible" (*arrhetoi*, as the irrational magnitudes were also called) into another linguistic system, hence appearing as incommensurable to it. Incommensurability, then, is fundamentally a way of showing the incapacity of a system to say all what is possible to say: there are things that can be said in some languages that cannot be said as well in others.
> (Chiurazzi, 2013: 43)

This definition of incommensurability fits Klein's idea of the difference between conceptualities. What is expressible through the formal language of algebra is not completely translatable into ordinary language. For Klein, the question does not concern the relation between languages *per se*, but rather the conceptualities which lie beneath them and determine the specific limits of each linguistic system. The asymmetry of the language, and consequently the friction we experience if we try to solve a mathematical problem with words instead of symbols, represents just the surface, the sign, which hints at the deeper difference which is constantly ignored. This sign is rather an *occasion*, what allows us to disengage ourselves from the yoke of modern conceptuality and gain access to another level of analysis. It offers the possibility of seeing differences that otherwise would remain hidden.

9.2.6 The method or the question of history

Language indicates the path, but history teaches how to follow it. As we said, Klein's historical attitude is *sui generis*. It could be defined as "genealogical." Although it is indeed realized by subtle and erudite philological analyses, the overall critical posture of the study prevents a certain "factual" interpretation of the analyses themselves. The research into the "origin" and the "conceptual structure" of the formal language does not amount to the investigation of the first "cause" that put it into being. In this second view, "the explanation of an idea becomes a kind of historical legend, a piece of anthropology" (*LE*, 66), in a word, just a story of accidental events. What is ultimately sought, instead, is something that belongs to the transcendental realm and, in this sense, it is formal. Although it is concealed *within* historical facts, it is not reducible to them.

This critical-genealogical attitude is what allows Klein to avoid the "naturalistic distortion" (66) attacked by Husserl and to open up the possibility of his major discoveries. Namely, it guarantees his detachment from some dogmas of our "modern consciousness" by implying (1) the suspension of the obvious validity of a supposed "*fact* of science" and correspondently, in the case at hand, "the *Faktum* of symbolic mathematics" (*MB*, 4), and (2) the disclosure of the possibility of seeing a different "how" where common understanding, granting the aforementioned "facts," cannot but see an identity or unexplainable ruptures. This second a-critical view, that we label as "anachronism," is reflected by some historical reconstructions (canonical at Klein's time), in which the contemporary situation of philosophical or mathematical knowledge is assumed to be the ultimate one, so that past events are read by projecting the shadow of the present on them. *Either* there is recognized a substantial difference between, for example, mathematical objects belonging, respectively, to Greek or modern science, and then history is viewed as a progressive realization of mental capacities from lower to higher abstractions, *or* no difference is noted at all, and then the question regards only the quantity of mathematical "information" a period is able to accumulate. "*Both* interpretations, however, start from the present-day condition of science" (*LE*, 9) and thus, they are affected by the same optical illusion. Opposed to this position, Klein claims that "our object is not to evaluate the revival of Greek mathematics in the sixteenth century in terms of its results retrospectively, but to rehearse the actual course of its genesis prospectively" (*MB*, 5).

According to Klein, the philosophical premises of the conceptions guiding these interpretations are completely wrong, and consequently they are incapable of giving a satisfactory account of the nature of mathematical objects. They start from the end and read the past as a preparation for the present. One might argue, nonetheless, that in pointing to an incommensurable difference between a Greek and a modern conceptuality, Klein seems to reverse the same perspective. If the "anachronists" go up to down, from the already justified present to the past, Klein's view follows the other way around, from the past to the present, with the risk of offering an axiological reading of the history of mathematics, from the primordial and genuine Greek period to the derivative one of modernity.[14]

Even though these criticisms are well grounded, they can be met with a reply within Klein's philosophical framework. Firstly, Klein is certainly aware that this bold categorization is, at first, just a useful tool for his analysis and nothing more:

> We have to recognize the possibility of a dangerous confusion similar to the one I mentioned with regard to Mathematical Physics. The results of historical investigations based on specific historical concepts and methods of interpretation ought not to be confused with the real picture of a real past. Not to see that, means to surround us with a pseudo-historical horizon of almost mythical quality so as to make us talk glibly of "Greek culture," "medieval times," "Renaissance," the "Seventeenth century," the "Age of Enlightenment," etc.
>
> (*LE*, 137)

The utility of this categorization as a tool is related *both* to the strategy and to the motivation of the investigation. That is, in order to understand the present-day difficulties stemming from modern rationalism, we have to compare our modern conceptuality with another one, and in this way trace it back to its origin and unresolved tensions. In this sense, *assuming* a "Greek conceptuality," and so other "pseudo-mythical notions" like "Greek mathematics," etc., is *at the beginning* a mere heuristic conceptual devise, a hypothesis which waits for a further and more precise validation. The choice is determined by Klein's "linguistic" strategy: since the symbolic apparatus is the essential outcome of modern conceptuality, and since "symbols did not exist for the Greeks" (*LE*, 43), the Greek non-symbolic conceptuality then represents the mirror of modernity. *At the end of his research*, this assumption is transformed into a *technical* notion whose characterization is drawn from the Scholastic distinction between *intentiones primae* and *secundae*. We shall see later the details and the reason for it; for the moment, we want to underline the fact that the notions of "Greek conceptuality" and alike are not arbitrary, or even mythological, historical hypothesis. They have

14 Acerbi (2010: 108–109) explicitly criticizes Klein's perspective as the *verso* of anachronism. According to him, both share a conception of history which reads ancient past mathematical facts by taking for granted the subsequent results.

at first the goal of *activating* the research.[15] They are also the *result* of the analysis which supports and specifies the original hypothesis. Thus, these distinctions cannot be attacked *in principle*, as a naïve philosophical premise, but exclusively *de facto*, i.e. considering Klein's historical and philological arguments.[16]

Secondly, the criticism that associates Klein's historical perspective with the "anachronistic" one seems to suggest that for him the development of formal sciences proceeds by linear steps. Just like there is a leap from a mathematical object to another one which is supposed to be "more abstract," the same goes from one conceptuality to the other. But this is not the case. On the contrary, Klein's difficulty "is precisely to avoid interpreting their differences and their affinity one-sidedly in terms of the new science" (*LE*, 9). His concern is not so much about differences in themselves but rather about the *relation* between them. Thus, primarily, he does not want to show the primitive genuineness of Greek science and then condemn contemporary science; rather, he tries to comprehend *our own* critical situation by means of the genealogical reconstruction of Greek conceptuality. In a word, the *Math Book* speaks more about the present than about the past. In this light, when we talk about "incommensurability," we do not simply stress a radical heterogeneity. We have to rule out the historical continuity-incommensurability dichotomy as a false one.[17] Saying that the two conceptualities are incommensurable with each other means to introduce into history a synthetical element, not a gulf which cannot be healed. A transformation would be simply impossible if the two were condemned to radical incommunicability. The conception of history is thus *dynamic* because it does not presuppose a linear progress, nor a series of inexplicable conceptual leaps resolved, thanks to the mythological power of "abstraction." It takes instead a "transcendental" perspective based on continuity, such that it can finally account for the *transformation* of the manner of conceiving and forming concepts.

In conclusion, Klein's critical perspective introduces the *possibility* of seeing a transcendental transformation in the basic concepts of mathematics. Hence, it can be traced only *transcendentally*, disregarding the objectivity of mathematical objects. As Hopkins claims,

> The knowledge of these different conceptualities and their philosophical significance presupposes a cognitive perspective whose goal is more the assessment of the *manner of cognizing* [my emphasis] the objects operative in the paradigmatic mathematical and philosophical cognition of the ancient Greeks and early moderns than in these objects themselves. Such a cognitive perspective is, following Kant, appropriately characterized as "transcendental", because its guiding interest is not in the mathematician and philosopher's knowledge per se but in knowing *how* the cognition operative in their kinds of knowledge comes about.
>
> (2014: 370)

15 "Klein thinks we should study ancient philosophy and the ancient life-world in order to make the modern conceptual frame [...] unfamiliar to us" (Manca, 2017: 48).

16 Regarding the history of Greek mathematics, the most complete critique of Klein's reconstruction can be found in Majolino (2012). Regarding the modern part, see Oaks (2018). In this work, we will not take into account their specific *historical* arguments.

17 As Chiurazzi argues: "[incommensurability] does not exclude, but *presupposes* relation: incommensurability *per se* is in fact a *contradictio in adjecto*" (2013: 43).

Thus, the incommensurable and occluded difference between Greek and modern conceptualities that Klein phenomenologically exhibits can be properly called a *transcendental difference*.

9.3 The static difference between *Arithmos* and *Zahl* and the critique of anachronism

9.3.1 Klein's specific thesis

Klein's exemplary study in the *Math Book* concerns therefore the transcendental transformation that the concept of number underwent in the modern age. As we have already argued, his ultimate interests are critically directed toward the disclosure of the fundamental presuppositions of modern conceptuality. Klein's decision to consider first of all the specific case of the concept of number is justified historically because symbolic language was first constituted in mathematics, and applied only afterward to other scientific fields. In this way, the analysis that engages the transformation of the understanding of the being of number becomes "exemplary" for what occurred in other epistemological domains, including the philosophical one. From his examination, in fact, he infers the formal structure of modern conceptuality in opposition to the ancient one, exposing the root of the fatal "danger" brought about by formalism.

Klein's specific thesis is that the *modern* concept of number is determined by a *transcendental* transformation that came about through Vieta's *historical* assimilation of Diophantus' logistic[18] at the end of the sixteenth century.[19] Namely, while for Greek mathematical and philosophical thought, a number (*arithmos* or *Anzahl*) is always a definite amount of definite items, for modern thought, it is instead the indeterminate correlate of a mark whose meaning is defined by a rule-governed calculus (*Zahl*). In Klein's account, Vieta's work represents the epochal threshold of the passage from the former to the latter because it introduces for the first time a fully algebraic language which, as Klein maintains, entails a symbolic conceptuality.

9.3.2 Static characterization of Arithmos

An *arithmos* or *Anzahl* is a definite amount of definite items. These items can be either things sensibly perceived or pure monads whose being is accessible only thanks to the intellect. Examples of *arithmoi* are "two chairs," "three tables," "five monads" and so on. This understanding is common to the entire Greek tradition and Klein holds that it survives in Europe just until the appearance of modern algebra. As Klein points out in chapter 6 of his *Math Book*, the meaning of this concept is determined by the ordinary practice of counting, or "counting-off" (*Abzählen*), things. An *arithmos* is nothing but a definite amount of things, i.e. things that are *counted* or listed one after the other according to a certain "idea." During this process, the things submitted to enumeration are homogenized: "These things, however different they may be, are taken as uniform when counted; they are, for example, either *apples*, or apples and pears which are counted as

18 The *Arithmetica* is the text of the Alexandrine mathematician composed in the third century AD and whose first six books survived and were rediscovered in the late fifteenth century in Europe.

19 Recent historical studies on Vieta are Oaks (2018), Panza (2006, 2007, 2010), Serfati (2005).

fruit, or apples, pears and plates which are counted as '*objects*'" (*MB*, 46). The enumeration comes to an end when the last "one" is listed, and the associated number is said: "That word which is pronounced last in counting off or numbering, gives the '*counting-number*' [*Anzahl*], the *arithmos* of the things involved" (46). Counting off means *saying* how many times an item of a certain type has been listed.

Having in mind the fundamental phenomenon of counting, we can understand why Klein recognizes a twofold *definiteness* in each *arithmos*: (1) "it is, first of all, a definite amount of objects *determined in such and such a way*" and (2) "it, secondly, indicates that there are *just so and so many* of these objects" (48). Thus, the twofold definiteness depends on which part of the definition above we want to stress: either we consider the *arithmos* insofar as it is always a number of (counted) things, i.e. of *definite items*, or insofar as it ends up always with a *definite amount* of these things (the enumeration ends necessarily with a last word which expresses the precise number of the group of things). The first aspect can be labeled as the "inseparable character" of the *arithmos* with respect to things. Just like in "the process of counting, in the *actus exercitus* (to use a scholastic terminology), it is only the *multitude of the counted things* which are in view" (49),[20] an *arithmos* cannot be severed from the precise multitude to which it belongs. Thus, "ten sheep" and "ten dogs," even though they are the same amount of things, are not the same *arithmos*. In this light, holding that "two and two makes four" is not trivial. If this proposition is considered "formally," that is, without explicating which things "two" and "four" are related to, it makes no sense. In fact, it can be false if we make it clear that we are adding, for instance, "two apples" and "two pears" to obtain "four apples," but it becomes true if we make clear that we obtain "four fruits."

The second aspect of the *arithmos* focuses on its being always a *definite amount* of things. In other words, there must be an end to the counting off of a certain group of objects. As a consequence, the definition excludes a series of "numbers" whose legitimate ontological status could have been accepted only later: zero, fractions, negative, variable and irrational numbers are not *arithmoi*. Not being a multiplicity, the "one" is also excluded, "a fact which seems strange only if we presuppose the notion of the 'natural number series'" (49). A "one" cannot be counted, even though it allows the counting of a certain multiplicity. In this sense, it assumes the specific status of "principle" of the *arithmoi*: "this is the case because it has the character of a 'beginning' or 'source' (*arché*) such as makes something of the nature of 'counting' originally possible" (49).

9.3.3 The two fundamental problems of the Arithmos

Despite it being derived from the apparently non-problematic practice of counting, Klein holds that "this concept of number involves two problems, two fundamental problems of Greek mathematics and philosophy." They correspond to the twofold structure of the *arithmos*. In fact, (1) since they are based on counting and therefore are inseparable from things, we should ask philosophically "what is the character of things insofar as they are counted? In what sense are they 'units' submitted to numeration?" and (2) since it is a certain definite amount of things, we should consider "in what sense is the number of those things or 'units' in itself a unity? Is the number expressed by *one* word a unity at all?" (*LE*, 45). According to Klein, the first

20 We quote Hopkins' translation in (2011: 177), which is slightly different from the English edition.

problem was addressed for the first time by Plato when he looked for a "foundation" (*Begründung*) of the practices of counting and calculating. In our common attitude toward the world, we always deal with numbers of ordinary things (apples, chairs…) with which we make ordinary calculations. Plato's question concerns what renders every ordinary thing a possible counted-thing. We said that an *arithmos* is composed of definite items: but what does that mean? How can a thing be considered as "one," i.e. what is its specific character insofar as it can be counted and submitted to calculation? It is a fact that we *do count* all different kinds of things, so that we have already at our disposal a certain understanding of "numbers" and "units" as such. At the same time, no sensible thing can provide the foundation for every particular enumeration. For example, "one apple" is not "one" because of its being a sensible apple, nor can it account for the being one of a pear – nonetheless, in the practice of counting apples, fruits or objects, it *can* be considered a definite item. This problem forces the philosopher to shift his gaze from particular and everchanging things toward "an object which has a purely noetic character, and which exhibits at the same time all the essential characteristics of the *countable* as such" (*MB*, 50). This "conversion" of the gaze leads to the opening of noetic realm of "pure" units whose supposition (*hypothesis*) makes possible ordinary calculation. Thus, the possibility of counting whatever sort of things "is exactly fulfilled by the 'pure' units, which are 'nonsensual,' accessible only to the understanding, indistinguishable from one another, and resistant to all partition" (50). These are the essential characters that determine a certain thing as a definite item, that is, *a thing insofar as it is counted*.

Thus, the supposition of this noetic field allows Plato to found and understand the process of homogenization that takes place in the practice of counting. It defines essentially this practice indifferently from the matter of the things counted: "to determine a number (*Anzahl*) means to count off in sequence the given single units, be they single objects of sense, single events within the soul, or single 'pure' units" (52). This is the case because the supposed pure units have those properties (homogeneity, external discreteness and internal indivisibility) belonging to every possible counted-thing. One apple or any other sensible thing is not homogeneous to the others, nor is it indivisible. However, in the practice of counting, it is both. It is so because the sensible apple is not the *actual* object to which the intellect is directed during the enumeration:

> the fact that we are able to count off a definite multitude of objects of sense is grounded in the existence of 'nonsensible' monads which can be joined together to form the number in question and which our thinking, our *dianoia*, *really* intends when it counts or calculates things.
>
> (71)

The definition of counting and thus of the *arithmos* is even indifferent to the origin and the mode of being of the monads, i.e. whether they are separate from sensible things or obtained through abstraction from them. It is clear that here we are pointing at Aristotle's critique of Plato's *chorismos* thesis – a critique that Klein interprets in an original way.[21] What Klein wants to stress, however, is that Aristotle's apparent opposition against his teacher concerns a different conception or doctrine of number and not a different understanding of its being, i.e. its conceptuality. Even for him, an

21 See Chapter 8 of the *Math Book*.

arithmos is a definite amount of definite things, be they sensible or pure items. The dogmatic difference can be viewed as an actual contrast only once it is acknowledged that they both a-critically presuppose the same conceptual dimension. There can be an argument about the *mode of being* of number (that is, as separate or not) only on the basis of a shared fundamental understanding of the *being* of numbers.[22]

The second fundamental problem concerns the possibility of the unity of a certain amount of things. An *arithmos* is always a multiplicity: what is it that assures that this multiplicity is at the same time a unity distinguished from other multiplicities? The question was once again taken into account by Plato following the Pythagorean tradition. The Pythagorean solution introduced the notion of *eidos* as something unitary that is such that it makes a unity out of a multiplicity. Every *arithmos*, in fact, belongs or participates to a certain *eidos*, in such a way that its being is determined by it. For example, since six things can be arranged yielding a triangle (just like three things or ten), which is a form, they are one insofar as they are a "triangle":

> Six is 'one,' or rather any six things can be conceived as *one* group, namely, 'six,' because the Form 'triangle,' which is one in itself, causes these six things to be one. All numbers under a certain Form belong to that Form exactly as all trees belong to the species 'Tree'.
>
> (*LE*, 47)

Every tree is what it is only inasmuch as it participates to the single Form "Tree" which defines it, i.e. limits its nature with respect to other things. Accordingly, every *arithmos* gains its definite nature from its unitary eidetic source.

> While the determination of each number as a "number of something" is given by the pure units or the given objects, the other aspect of the determinateness of a number, namely the fact that it is always a *definite* number (of pure units or of things of some sort), can be understood only as the consequence of the special *kind* to which it belongs, i.e., by means of *which is in itself one* and is thus capable of unifying, of making wholes – of delimiting.[23]
>
> (*MB*, 56)

22 Hopkins remarks that Klein's rediscovery of the concept of *arithmos* in Greek thought

> does not amount to the claim that 'the Greeks' possessed a univocal understanding of this concept. [...]. Rather, Klein's claim is that the various accounts presented by the Greek mathematicians and philosophers of the mode of being belonging to the pure units that compose *arithmoi* or of the kind of priority belonging to the unit with respect to an *arithmos* all 'stem from one and the same original intuition, one oriented to the phenomenon of counting'.
>
> (2011: 180)

23 Klein remarks that the Aristotelian critique of this second solution reduces the ontological significance of the Platonic *eide*. Since, for Aristotle, "a number is simply not *one* thing but a 'heap'" (*MB*, 107), the *eidos* of a certain number does not assure its unity, but it is rather a *property* of it, an instrument that allows it to be catalogued. This ontologically weaker notion will be mathematically inherited by Diophantus and then transformed in a symbolic way. The process is continuous:

> When we look back to the Pythagorean and Platonic concept of the *eidos* of an *arithmos* as that which first makes the unified being of each number possible [...], we may say that the *ontological* independence of the *eidos*, having taken a detour through the *instrumental* use made of it by Diophantus, finally arrives at its *symbolic* realization.
>
> (*MB*, 175)

9.3.4 Static characterization of Zahl

A *Zahl*, the modern concept of number historically established in Vieta's *symbolic* use of *eide* in the *Isagoge* (1591), is defined as the general *concept* of a multitude, whose being is inseparable from the concrete mark written down and symbolically associated with it, be it "0," "4," "–5" or the variable "x." In contrast to the direct relation of *arithmoi* to the ordinary practice of counting, the modern number is neither derived directly from counting, nor from a theoretical consideration of sensible multiplicities given in perception. Its grounding phenomenon is rather the usage of the Arabic positional system of ciphers introduced in Europe during the twelfth century, "whose 'sign' character is much more pronounced than that of the Greek or Roman notation" (*MB*, 277).[24] This new system of numeration "leads to a kind of indirect understanding of numbers and ultimately to the substitution of the ideal numerical entities, as intended in all Greek arithmetic, by their symbolic expression" (*LE*, 83). It promotes, in other words, a prominent "blind" usage of concrete marks whose signification can be fruitfully ignored by an algorithmic calculus and possibly restored only at the end of the process. The specifically mechanical character of the calculus can be highlighted once we contrast it to ordinary language: to calculate is to play a formal game in silence, without words, which is a practice quite distinct from "thinking" in the Platonic sense of *dianoia*, that is, as "a voiceless dialogue of the soul with itself" (*Sophist*, 263 E). Strictly speaking, during the calculus, there is nothing to say and thus nothing to "think," as what is written functions independently of the voice and the soul. In this way, the "meaning" has been somehow transferred from the internal receptacle of the mind to the external one of the writing itself.[25]

These words summarize Klein's entire philosophy of the history of mathematics. It is time now to consider carefully the nature of this "symbolic realization."

24 However, Klein adds, quite ambiguously, that

> it would be a mistake to attempt to understand the origin of the language of symbolic formalism as the final consequence of the introduction of the Arabic sign language. The acceptance of this sign language in the West *itself presupposes a gradual change in the understanding of number*, whose ultimate roots lie too deep for discussion in this study.
>
> (*MB*, 277)

25 During the period when Klein was working for his articles, authors with diverse philosophical backgrounds put into question the algebraic thought in these terms. For instance, Alfred North Whitehead noticed lucidly:

> these symbols are different to those of ordinary language, because the manipulation of the algebraical symbols does your reasoning for you, provided that you keep to the algebraic rules. This is not the case with ordinary language. You can never forget the meaning of language, and trust to mere syntax to help you out.
>
> (1927: 2)

Analogously, Simone Weil wrote in 1934:

> Le calcul met les signes en rapport sur le papier, sans que les objets signifiés soient en rapport dans l'esprit; de sorte que la question même de la signification des signes finit par ne plus rien vouloir dire. On se trouve ainsi avoir résolu un problème par une sorte de magie, sans que l'esprit ait mis en rapport les données et la solution.
>
> (Calculation puts signs in relation on paper, without the signified objects being consciously related; so that the very question of the meaning of the signs ends up meaning nothing. We thus find ourselves having solved a problem by a kind of magic, without the mind having linked the data and the solution.)
>
> (1955: 67)

Like the *arithmos*, the *Zahl* has a twofold structure, but one that is incommensurable. *Zahlen* are defined by Klein as "*formations whose merely possible objectivity is understood as an actual objectivity*" (175). We have here the combination of an indeterminate aspect (possible objectivity as potential determination) with a determinate one (actual objectivity). Thus, a *Zahl* is, on the one hand, (1) a general or indeterminate *concept* of a possible multitude (for example, the general concept "twoness" or "natural number"), on the other, (2) a concrete *mark* which allows its symbolic representation (for example, mark "2" or "n"). The paradoxical union of these two aspects determines the symbolic being of the modern number.

The first aspect explicitly negates the "inseparable character" of the *arithmos*. In fact, a *Zahl* is not primarily a number of something, be it sensible things or pure monads. Being a general concept and not a certain multitude of things, it is detached from anything whatsoever. Taken abstractedly, it has no "body." An algebraic symbol like

> 'a' *can* be 4, or 6, or 150, or any possible number, but 'a' *is not* 4, or 6, or 150, or any other number. Thus, 'a' doesn't mean certain objects, namely units, the multitude of which it indicates, but rather the *concept* of the number as a multitude of units.
>
> (*LE*, 62–63)[26]

This means that the "general" concept is deprived of a "look" (*eidos*) and therefore is literally unimaginable. This impossibility does not hold for Greek numbers. For example, the *arithmos* "four" *can* be imagined as a group of four points yielding a square: even though the noetic group of four monads is veritably accessible only through the effort of the intellect, the particular four points drawn on the paper are a sensible *image* of them and arguably an essential condition to achieve scientific knowledge of them. The same can be said of geometric figures with respect to diagrams.[27] But how could we have a "copy" of concepts like "twoness" or "natural number"? It is a fact, nonetheless, that we *do represent* these concepts in a certain way, namely through marks "2" and "n." What is then the (symbolic) relation between the determinate mark and the general number? For sure, the algebraic mark is not a "sign," if we think of this term in the traditional sense expressed by the formula "*aliquid stat pro aliquo*." It is not a sign because there is not a definite object that can be expressed precisely. Rather than making a certain object intellectually comprehensible, the mark makes it concretely accessible by making possible its *manipulation*. This is the second aspect, the "reifying" one, of the mathematical symbol, i.e. it being taken as "an actual objectivity" despite its "generality." Since the mark represents the concept in its own body, the two are practically, i.e. during the practice of manipulation, indistinguishable.

> For instance, "2" does not signify something other than itself, for example, the exact amount of some kind of object; instead, it presents itself as the "concept

26 "The transformation moves from the unity of determinate amounts of the units composing *actual* multitudes to the *concepts* as such of *possible* multitudes" (Hopkins, 2016: 544).

27 The classical reference is to *Republic*, VI, 510d:

> And do you not also know that they [geometers] further make use of the visible forms and talk about them, though they are not thinking of them but of those things of which they are a likeness, pursuing their inquiry for the sake of the square as such and the diagonal as such, and not for the sake of the image of it which they draw?

of two," which means, "the general concept of twoness in general" – and it does so in a manner that involves absolutely no immediate reference to any individual things.

(Hopkins, 2012: 292)

That is, there is no distinction akin to the case concerning sensible image and intelligible model, but rather a *collapse* of the indeterminate and unintelligible concept into the sensible mark. These two moments can be phenomenologically distinguished, but they belong to the same formation: there is not a concept on the one hand and a mark on the other – there is just *one* symbolic being whose structure is twofold.

Needless to say, it is not sufficient to write single letters or ciphers in order to constitute symbolic formations. Letters *per se* are meaningless figures. The point is rather how they are to be conceived and used, alphabetically or algebraically.[28] The general concept becomes determinate in the concrete mark insofar as it becomes subject to certain syntactical rules. Since the former is divested of any "essence" (*eidos*), that is, it is completely "formalized" and literally unintelligible through *logos*, it nevertheless gains a *meaning* by virtue of the stipulation of rules which determine its behavior within a calculus.[29] In effect, "it is precisely the character of its mode of being as 'unintelligible' that necessitates its involvement with 'meaning,' namely, with the meaning that accrues to it on the basis of the 'stipulation' of rules for manipulating otherwise 'unintelligible' sense-perceptible marks" (293). The general concept is taken nonetheless as a "thing" when the mark is implanted in a system of calculus: it is the *concrete mark* ("1" or "x") that is accessible to manipulation according to purely syntactical operations.

We can notice at this point that the two fundamental problems of the *arithmos* are now meaningless. It is the definition of number as definite amount of definite things which requires an eidetic "foundation," external to the strictly mathematical domain, such that it guarantees the generality of the procedure to any object having the same *eidos*. In modern conceptuality, instead, objects are not thought in accordance with their universal *eidos* but *are defined by the procedure itself*: "modern mathematics [...] turns its attention first and last to *method as such*. It determines its objects *by reflecting on the way in which these objects become accessible through a general method*" (*MB*, 123). Mathematics has gained a decisive independence from any *ontological* foundation because of its self-sufficiently exact (functional) definition of its objects.[30]

28 Talking about the establishment of Vieta's *logistice speciosa*, i.e. the calculus conducted only with letters, Klein remarks that it gives "the capacity not only of *writing* an expression such as ax + b = c [...] but also of *calculating* [my emphasis] with this expression" (*LE*, 25).

29 Geometrical figures, after Cartesian analytic geometry, will share the same destiny. See Klein (1985: 12-21), Lachterman (1989) and Ferrarin (2014).

30 "[I]l pensiero antico necessita di un piano ontologico che consenta di considerare i risultati ottenuti su un caso particolare come validi per tutti i casi possibili: l'interpretazione ontologica risulta così centrale nel processo di assicurazione dei procedimenti operativi della matematica greca. [...]. Al contrario, con Viète e con il passaggio alla matematica simbolica, è la stessa prassi dei matematici a *determinare inconsapevolmente* il livello ontologico dei suoi oggetti. *La matematica si sostituisce alla filosofia come forma di sapere determinante il modo d'essere delle entità in gioco nei suoi procedimenti*" (Ancient thought requires an ontological basis that allows the results obtained on a particular case to be considered as valid for all possible cases: the ontological interpretation is thus

9.3.5 A Kleinean example

The previous characterization of the difference between *arithmoi* and *Zahlen* can be clarified with an example taken from a lecture entitled “The World of Physics and the ‘Natural’ World” that Klein gave in 1932. Consider the following sentences (*LE*, 23):

1 ς^{οἰ}β Μγ ἴσοι εἰσὶν Μσι ζ
2 2x + 3 = 7

(1) is a mathematical expression that appears in the Diophantine *corpus*. Despite the appearances, it does not contain symbols, but only *abbreviations* of words derived from ordinary language. In particular, the first letter, “ς,” is a ligature for *arithmos* (ἀριθμός) and it refers to the unknown magnitudes that are to be discovered. As Klein shows, the sentence can be easily transformed into an explicit grammatically correct sentence in Greek, thereby it can eventually be translated into modern English as “two numbers [*arithmoi*] and three monads are equal to seven monads.” (2) is instead the translation of (1) in contemporary algebraic language, so that “x” is the modern counterpart of the ancient “ς.” Klein asks: “Is this merely a technically more convenient form of writing? Do the two equations say entirely the same thing, if we disregard the mode of writing?” (23). As we already know, the answer is *no*. The abstract operation of separating the linguistic form and the mathematical content is unwarranted in a critical perspective. On the contrary, relying upon the previous analysis, we can now figure out some differences between the two ways of thought carried by the *verbal* and the *symbolic* expressions.

Three distinctive elements can be pointed out regarding the sign “ς” and the symbol “x.”

1 Although they are both letters of the (Greek or Latin) alphabet, their roles are radically distinct. Especially, the relationship with the voice, still essential for the meaning of “ς,” insofar as it is a case of “*economie verbale*,”[31] is definitely severed by the symbol “x.” It is possible to understand the sign “ς” only if it is known that it stands for the *word* “*arithmos*.” A symbol, instead, is not dependent on the eidetic meaning gathered by the inner voice of the soul, but on the external operational one. One can of course “read” a formula, but this is a different practice from reading a sentence in ordinary language: the latter is *originally* intended as having this scope, like a musical score.
2 “ς” is always related to a certain amount of items. The fact that the letter “ς” or the word “ἀριθμός” replaces a quantity whose number cannot be said at the beginning of the calculation allows it to be treated *as if* it was already given. The sign, in this way, refers to a certain number of monads although it is still hidden. The natural assumption under this practice is that one always calculates with

central in the process of ensuring the operational procedures of Greek mathematics [...]. On the contrary, with Viète and with the transition to symbolic mathematics, it is the practice of mathematicians *to unknowingly determine* the ontological level of its objects. *Mathematics replaces philosophy as a form of knowledge that determines the way of being of the entities involved in its processes*.) (Chiaravalli, 2018: 29).

31 See Cobb-Stevens (1997: 90).

arithmoi, so that "the multitude of monads which the unknowns number contains is indeterminate only 'for us'" (*MB*, 140). That is to say, it is not objectively indeterminate, but its indeterminateness depends upon our ignorance.[32] Therefore, "the unknown is to be understood as an 'indeterminate multitude' only from the point of view of the solution, namely as 'provisionally indeterminate,' and as a number *which is about to be exactly determined in its multitude*" (140). If the calculation provides a solution whose "number" is negative or irrational, the entire problem is rejected as "impossible," according to the limiting conditions stated at the beginning of it. In the end, the *eidos* in this context is employed to talk about and calculate with an *arithmos* even though it is still unknown. Having lost the original ontological significance it had in Platonic philosophy, it preserves the role of "*mere property* of the various numbers," as a valuable principle of cataloguing. The mathematician knows in advance the type of number that is being looked for.[33] On the contrary, "x" *is* an indeterminate number; it is treated as such symbolically. (2) is a formula; thus, it does not necessarily require that "x" expresses a precise "amount of monads." It can remain indeterminate and, and in order to reach the "generality" typical of scientific propositions, it *should* remain so. Since (2) also deals with determinate numbers, and so the "generality" achieved is still not the highest, "from the point of view of modern algebra only a single additional step seems necessary to perfect Diophantine logistic: the thoroughgoing substitution of 'general' numerical expressions for the 'determinate numbers,'" the step finally taken by Vieta (*MB*, 139). The transformation of (2) into "$ax + b = c$" is possible only once the marks in (2) are already thought as *symbols*, i.e. as indeterminate numbers whose possible determination with numerical values is understood somehow as an actual objectivation. Or, put differently, symbols are indeterminate numbers whose need for fulfillment with determinate values can be suspended, thanks to the reification operated through the mark.[34]

32 An illuminating analogy comes from ninth-century Islamic theology, contemporary to al-Khwarizmi's work that initiated the Arabic algebraic tradition. In his *Kitab al-jabr wa al-muqabala*, he employs the term *al-shay'* to refer to the unknown in an equation; in the theology of his time, this term attributes to God an assured existence whose *knowledge* is nonetheless indeterminate (Catastini et al., 2016: 10).

33 "In sostanza, gli *eide* consentono di parlare di un *arithmos* anche quando la sua quantità non sia ancora stata definita. Il 'numero ignoto' viene identificato in base alla categoria in cui può essere classificato: di quell'*arithmos* io, matematico, so che è un 'numero quadrato'. Ciò implica a sua volta che le forme platoniche, gli *eide*, si sono trasformate da unità ontologiche che garantiscono l'identità numerica in mezzi tramite cui il linguaggio matematico si esprime sui propri oggetti. L'*eidos* è ormai divenuto uno strumento" (In essence, the *eide* allow us to speak of an *arithmos* even when its quantity has not yet been defined. The 'unknown number' is identified on the basis of the category in which it can be classified: of that *arithmos* I, the mathematician, know it is a 'square number.' This, in turn, implies that the Platonic forms, the *eide*, have been transformed from ontological units that guarantee numerical identity into the means by which mathematical language expresses itself in relation to its objects. The *eidos* has now become an instrument.) (Chiaravalli, 2018: 27).

34 The distinction is explained clearly by Oaks, when he says that in premodern algebra, letters are kinds or denominations and so they are to be necessarily accompanied by a numerical "coefficient." While "x" is already a number, letters like "ς" are meaningless, if they remain alone. The "coefficient" says how many the numbers of that kind are. Writing "1A + 6C" in premodern algebra equals to write something like "1 euro + 6 dollars." The expression "euro + dollar" or "A + C" would be patently nonsensical (2018: 269).

3 A corollary of the second point is that, in the case of Diophantus, "there is no sharp boundary between 'the determinate' and the indeterminate' numbers" (142). Properly speaking, there are no "indeterminate numbers" in Diophantine logistic, but only assemblages of monads whose precise number is still unknown for the mathematician. On the contrary, in the modern context, albeit both reduced to the symbolic level, indeterminate and determinate numbers are clearly distinguished. The passage from the former to the latter reflects the passage from the generality of the law to the particular existence of individuals, from the ideal realm to reality. The two moments are thus interrelated but opposed. On this ground, Gottlob Frege could set the criterion for the *absolute* ontological distinction between objects (like determinate numbers) and functions (expressed by formulae containing "x"). Since the latter are essentially unsaturated, i.e. incomplete because of the indeterminateness introduced by the variable, they "differ fundamentally from [determinate] numbers" (Frege, 1960: 133). Clearly derived from the symbolic transformation, this distinction would make no sense in a pre-symbolic conceptuality.[35]

9.3.6 *Anachronistic accounts of Vieta's innovation*

According to Klein, the modern concept of number is first established in Vieta's *logistice speciosa*, the first literal calculus with what he calls *species*.[36] His work represents the threshold of the transformation of the pre-symbolic understanding of number into the symbolic one. Historians of mathematics have always recognized Vieta's novelty. However, their interpretations have been generally affected by the anachronistic perspective criticized by Klein. As reported by Oaks (2018), Klein was the first scholar to contest the anachronistic prejudice and to offer a different perspective on this matter. Following the previous categorization, we can distinguish between a radical anachronism and a moderate one. Those who do not acknowledge any change in the nature

35 As Cobb-Stevens remarks:

> Les équations de Descartes sont des structures indéterminées définies par des relations plutôt que des objets définis par des prédicats. Cette préférence pour les relations caractérise aussi l'interprétation algébrique par Frege des propositions et de la logique propositionelle. Frege envisage les concepts objectifs par les prédicats comme des fonctions […]. Cette approche élimine en fait le rôle des *eidè*, c'est-à-dire les corrélats des prédicats. […]. Il s'ensuit que les cas particuliers considérés comme candidats aux places vides indiquées par les variables ne peuvent plus être considérés comme participant déjà aux caractéristiques spécifiques exprimées par les prédicats pertinents.
>
> (Descartes' equations are indeterminate structures defined by relationships rather than objects defined by predicates. This preference for relationships also characterizes Frege's algebraic interpretation of propositions and propositional logic. Frege sees objective concepts mediated by predicates as functions […]. This approach actually eliminates the role of *eide*, that is, the correlates of predicates. […]. It follows that the special cases considered as candidates for the empty places indicated by the variables can no longer be considered as already participating in the specific characteristics expressed by the relevant predicates.)
>
> (1997: 102)

Therefore, the relation between individuals and universals, essential within the Aristotelian paradigm, is broken and the two are left with their (symbolic) irreducibility. The issue of "participation" of the individual to the idea is thus transferred, within an extensionalist paradigm, into the question of "membership."

36 "As soon as 'general number' is conceived of and represented in the medium of species as an 'object' in itself, that is, symbolically, the modern concept of 'number' is born" (*MB*, 175).

of mathematical objects belong to the first category, while those who do acknowledge it but that are unable to conceive of it in an untrivial way belong to the second one. Radical anachronists interpret Vieta's mathematical innovation in terms of "generality." Vieta accomplished the substitution of ciphers for letters: he introduced literal coefficients in place of numerical ones (like from "2x" to "ax"). In the *Isagoge*, the crucial passage in this regard is the following:

> let the given magnitudes be distinguished from the undetermined unknowns by a constant, everlasting and very clear symbol, as, for instance, by designating the unknown magnitude by means of the letter A or some other vowel E, I, O, U, or Y, and the given magnitudes by means of the letters B, G, and D or the other consonants.[37]
>
> (1968: 340)

While Diophantus' "algebra" still deals only with numerical equations (and it is thus called *logistice numerosa*), the story goes, modern algebra (as *logistice speciosa*) finally takes the path of generality: it does not consider this or that particular numerical case, but it embraces all of them at once in a systematic way. This is possible once not only unknowns but also even given numbers in a problem are treated with letters.[38] According to this reading, there is not a deep difference between Diophantus' and Vieta's "algebras." "Equations" in Diophantus and their symbolic counterpart in modern algebra concerned the same type of objects – namely, numbers (*Zahlen*). Stated simply, Diophantus always considers *specific* numbers; Vieta also considers general cases, to which specific cases are "logically" subordinated.

Nonetheless, the passage to generality, that is, the substitution of the cipher with a letter, seems to be obvious. At first sight, it seems that the particular solutions of equations *presuppose* the general solution. A numerical equation is solved correctly because its specific resolution instantiates the general algorithmic procedure of resolution expressed by a formula. This raises a curious problem. How did Diophantus miss it? Why were the Greeks unable to make such a simple step toward generalization? The riddle becomes even more obscure if we think of Aristotle's use of letters in his logical and ethical treatises, where they seem to have, from a modern perspective, the same role algebraic symbols have in formulae. Vieta and Descartes even hypothesized that Diophantus hid the *logistice speciosa* just to show his ability to solve "numerical" problems in a non-mechanical way.[39] This idea incredibly survived throughout

37 In present-day algebraic notation, we employ the letters at the end of the alphabet to refer to unknowns (x, y, z), while the letters at the beginning of it refer to given magnitudes (a, b, c…).

38 "In traditional numerical algebra the givens are assigned at the start, so the equation must be set up and simplified for each application. In *logistice speciosa* this work is done once with literal coefficients. With a simplified equation or proportion expressed in species the values of the unknown can be calculated directly for as many different sets of knowns as one wants" (Oaks, 2018: 271).

39 Vieta claims:

> Diophantus in those books which concern arithmetic employed zetetics [calculus with species] most subtly of all. But he presented it as if established by means of numbers and also by species (which, nevertheless, he used), in order that his subtlety and skill might be more admired.
>
> (1968: 345)

centuries and is still alive nowadays in some historical reconstructions. This demonstrates how strong and pervasive what Klein has called the "*Faktum* of symbolic mathematics" is, and sheds some light on the reason why his work has been neglected for a long time. The mistake consists in taking the passage from the determinate value to the indeterminate one simply as "logically required." Klein explains that it appears so only having presupposed the modern concept of number, so that, for instance,

> 'two' no longer means in Vieta 'two definite things,' but the general *concept* of twoness in general. [...]. It no longer means or intends a determinate number of things, but the general number-character of this one number, while the symbol 'a' represents the general numerical character of each and every number.
>
> (*LE*, 25)

They share, as it were, the same symbolic nature; thus, they can be easily substituted one to the other.[40] In other words, starting from Vieta, all "numbers" have been transferred to the symbolic level. Therefore, "numbers" in Diophantus (which are, in fact, *arithmoi*) are interpreted from the *already symbolic* conceptual level of modernity as *Zahlen*. Because the *ontological* difference between them is occluded, moderns are forced to explain away the *factual-historical* differences they have to admit (e.g. the absence of a literal algebra in Greek mathematics) through the boldest hypotheses. Thus, for instance, it is said that Greeks, consciously or unconsciously, already thought algebraically but somehow hid that reasoning, which is supposedly entailed by the unchanging "matter" of mathematical science.

Alternatively, there are moderate anachronists,[41] who have recognized a difference in the *nature* of the mathematical object "number" throughout history, especially in Greek thought and modernity. However, being affected by the same optical mistake, they interpret that difference having as measure the modern symbolic number. As a consequence, the Greek *arithmos*, being always related to a certain number of things, appears "more concrete" with respect to the purely "abstract" *Zahl*. This concreteness can be otherwise characterized mythologically as "intuitive" or "archaic," either to praise or to deplore it, introducing an axiological element which Klein, as we have

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Similarly, Descartes:

> For they perhaps fear, *just as many inventors* have been found in the case of their discoveries, that because the true mathematics was *easy* and *simple* it would become cheapened in becoming *popularized*, and they preferred to exhibit to us in its place *as the results of their art* certain sterile truths, very acutely demonstrated by deduction, so that we might admire those, instead of *the art itself*, which would have quite subverted our admiration.
>
> (*Regulae*, Rule IV)

40 As Chiaravalli notes: "La somma 1 + 3 = 4 non ci appare qualitativamente differente da x + y = z. A noi moderni sembra solo che la seconda somma sia la struttura generale di cui la prima è un esempio" (The sum 1 + 3 = 4 does not appear to us to be qualitatively different from x + y = z. It seems that the sum 1 + 3 = 4 does not appear to us qualitatively different from x + y = z. It seems to us moderns that the second sum is the general structure of which the first is an example.) (2018: 31).

41 Klein mentions the philosophers Julius Stenzel and Oskar Becker (*MB*, 63); the former is the author of the book *Zahl und Gestalt bei Plato und Aristoteles* (1924), whereas the latter is known for *Mathematische Existenz. Untersuchungen zur Logik und Ontologie mathematischer Phänomene* (1927) and *Die diairetische Erzeugung der platonischen Idealzahlen*, originally published in 1931 (English translation: *The Diairetic Generation of Platonic Ideal Numbers*, 2007).

already argued, tries to avoid.[42] What is uncritically presupposed in such accounts is not the object "number" as *Zahl*, as in the previous case, but rather the subjective way of encountering it. Since the mathematical conceptuality is taken as non-changing, the differences between mathematical objects appear just "ontically." That is, since different concepts of number are supposed to lie on the Procustean bed shaped by the same conceptual ground, they can be distinguished only by degree, according to "the traditional Aristotelian measure of abstractness, which is calibrated in terms of a concept's degree of remoteness from or proximity to sensuously perceived things" (Hopkins, 2011: 255). They look in this way as more or less "abstract" and more or less "concrete." Despite the superficial clarity of these words, their meaning remains undetermined, producing conceptual confusion. From Klein's standpoint, the appeal to "abstractness" is a false path to grasp the aforementioned difference, which is due to an insufficient understanding of the modern concept of number. In truth, "this facile and easily misunderstood manner of speaking leaves its true and complicated structure completely in the dark" (*MB*, 175–176). The alleged explanation is not only apparent since it is a case of an *obscurum per obscurius* argument, but it is also deceptive insofar as it exchanges the actual transcendental difference with an ontic one.

Klein's alternative transcendental perspective, explicitly opposed to the ontic one, is clearly stated in the following passage:

> The peculiarity of the Greek concept of "number" lies less in an "archaic" or "intuitive" character (which is not at all its property) than in *the kind of relation* it has to the "thing" it intends. [...] this relation of concept and intended content is subjected to a fundamental modification under the aegis of a new *Begrifflichkeit*. (63)

9.4 The dynamic structure of Klein's account of the symbolic transformation

9.4.1 Klein's dilemma

We want to finally set forth Klein's original reading of the transformation of the understanding of the being of numbers in opposition to the previous unsatisfactory anachronistic accounts. The disclosing of its internal dynamic structure is possible because Klein, instead of looking at the nature of numbers themselves, considers the way in which they are conceived of in language and how their concepts are formed. In the essay *Modern Rationalism*, he takes the following sentences as a basis for his reflections (*LE*, 61):

1 Five horses and six horses make eleven horses,
2 Five unknown quantities and the number six equal the number sixteen,
3 $ax + b = c$.

42 "[O]ne fundamental objection is to be raised immediately against stressing the 'intuitive' character of the *arithmos* concept, namely that it arises from a point of view whose criteria are taken not from Greek, but from modern, symbolic, mathematics" (*MB*, 63).

Now, it is patent that through this example Klein intends to highlight what are in his mind the crucial steps which illustrate the historicity of numbers. While (1) is an expression of an ordinary calculation with assemblages of sensible things and (2) a theoretical calculation with numbers of pure units, (3) represents instead a fully symbolic expression established only in modern times by Vieta.[43] The passage from (1) to (2) can be called the "Platonic step," since it attests to the opening of the noetic realm of monads and the possibility of a theoretical mathematical discipline. Klein claims that (2) is "an equation we can find in the textbook of Diophantus" (62). As Klein has explained, what distinguishes the two first numerical sentences is not the essential structure of numbers, which are in both cases a definite collection of items, but rather the *genre* of the objects counted. In (1), numbers are collection of sensible things (i.e. horses), whereas in (2), they are collections of pure units recollected under a specific *eidos* (e.g. the number six). Therefore, the passage from the first to the second is *linear*, that is to say, it goes "from concrete numbers to abstract numbers, since the arithmetical number *six* is abstracted from any possible group of six objects" (62). In this context, the Aristotelian way of abstraction is still capable of clarifying the nature of the theoretical passage.

The case of (3) is quite distinct. Letters here are *symbols* and do not determine directly through their meaning a certain amount of things. They rather denote the concept of a multiplicity, its mere possibility of assuming a numerical value. As the preceding discussion has demonstrated, the transition from "five" to the coefficient "a" cannot be explained away by employing the Aristotelian measure of abstraction, as if "a" was a super-number even more general than those composed of sensibly unperceivable units. Indeterminate numbers like "a" cannot be obtained by disregarding further traits from the ordinary collections given in sensible or intellectual experience. Strictly speaking, *every* trait has been abstracted, that is, "lifted off," from multiplicities: what is left (or rather imaginatively produced)[44] is just the naked mark, the empty shell of symbolism.[45] The last step involves a *non-linear* process of formation, i.e. a conceptual process that Klein calls "symbolic abstraction" and such that it involves an actual *transformation* of the way of forming concepts along with their relationship with experience.

Our goal now is to follow the path that allowed Klein to discover this peculiar kind of "abstraction"[46] covered at the junction of ancient and modern conceptualities. Having put aside the easy canonical "Aristotelian" explanation, Klein is thus committed to give another and more convincing hypothesis for the formation of (previously inconceivable) sentences like (3). In the simplest way, we could say that Klein's question is the following: how is it possible to substitute the word "five" with the

43 We do not take into account here the differences between contemporary notation and the actual Viètian text – differences that are, according to Klein, accidental and therefore irrelevant for the sake of the argument.

44 On the role of imagination in symbolic reasoning, see the part devoted to Descartes in the *Math Book* (§ 12, B).

45 "Whereas in the Greek conception, number is inseparable from a multitude of items to which it, as number, is directly related, in the modern conception number no longer relates directly to a multitude but to the *concept* of a multitude from which everything has been abstracted *save for a sign* [my emphasis] whose apprehension is identical with this concept itself" (Hopkins, 2014: 363).

46 Because this expression is misleading – since it seems to be related to the Aristotelian type of abstraction - in *Modern Rationalism* (*LE*, 62), he prefers to define this theorical operation as "generalization."

indeterminate coefficient "a"? The sense of the question, however, is genealogical, that is to say, it prevents the conception of the latter as a conceptual formation already presupposed at the beginning. A better restatement of the same problem would be this one: how do letters acquire a numerical significance despite their original alphabetical, and thus patently non-quantitative, nature? Or, to put it transcendentally: since it is a fact that we *calculate* with letters and other marks, how is it possible that these very letters *become* symbolic numbers? In other words, if Klein wants to give "an account of the condition of possibility of symbolic reason," he has to account "for precisely this *fact*, namely, that letters with otherwise no significant connection with mathematical objects are nevertheless recognized as unambiguously having such a connection" (Hopkins, 2014: 382).

9.4.2 Vieta's ambiguity

Being at the intersection of the unresolved tensions of modernity, Vieta, on Klein's view, cannot be but an ambiguous figure. His ambiguity consists in preserving an original source and, at the same time, opening up a new conceptual realm that eventually produced the symbolistic oblivion of that very source. In the case of algebra, the authentic source is the concept of number as *arithmos*, whose being is directly related to a group of things. Vieta's algebra is ambiguous because, while maintaining this original connection to reality, it *de facto* severs it. What is still audible in it and coexists with the derivative symbolic concept of number will be forgotten and completely superseded by the latter. The history of number is, then, a story of forgetfulness:

> In Vieta's notion of 'species' the original understanding of number is retained, as it is, of course, in the *Arithmetic* of Diophantus. But his immediate successors [...] have already lost the original intuition. The technique of operating with symbols replaces the science of numbers.
>
> (*LE*, 81)

In sum, Vieta is at the threshold of modernity insofar as another understanding of number is surreptitiously introduced and put alongside the Greek one through the introduction of the calculus with species. This understanding will eventually become the canonical one and the ambiguity still present in Vieta will be lost.

Klein's demonstration, then, has its basis on two pillars. It has to prove the two faces of the ambiguity, i.e. that *both* concepts of numbers (as outlined in the second chapter) implicitly coexist in Vieta's thought, and to figure out their mutual relationship. Klein shows that Vieta conceives of numbers as referred to determinate collections of determinate items through the interpretation of the so-called "law of homogeneity" (*lex homogeneorum*) stated in chapter III of the *Isagoge*.[47] This fundamental principle

47 "The supreme and everlasting law of equations or proportions, which is called the law of homogeneity because it is conceived with respect to homogeneous magnitudes, is this: Only homogeneous magnitudes are to be compared with one another" (Vieta, 1968: 324).

> says, in modern terminology, that all numbers in an equation must have the same dimension. According to it, only such magnitudes can be related by the way of addition or subtraction as belong to the same or the corresponding 'rung' [dimension], although this does not hold for multiplication and division.
>
> (*MB*, 173)

The principle, then, "is concerned with the fundamental fact that every 'calculation', since it does, after all, ultimately depend on 'counting off' the basic units, presupposes a field of *homogeneous monads*" (173–174). This law, at the basis of every specific calculational rule, is therefore the necessary background that grounds the ultimate significance of formal calculus. It was not explicitly stated by Diophantus in his logistic since "this demand is fulfilled as a matter of course, because it already operates within such a field of 'pure' monads" (174). Having transferred the calculation entirely into the domain of the indeterminate, the remark is now required, and it shows that "the concept of species is for Vieta, its universality notwithstanding, irrevocably dependent on the concept of *arithmos*" (174).[48]

This is the first side of Vieta's ambiguity and one of the two strands that determines his role in the transformation of the concept of number. So far, Klein has maintained that the heritage of the Greek conceptuality of number is nevertheless the natural background on which Vieta can found a symbolic language. The other side concerns the special kind of "universality" or "generality" that this concept assumes, i.e. its proper symbolic transformation. In fact, "the *character of arithmos as a 'definite amount of…' is preserved in it* [i.e. in Vieta's concept of species] *in a peculiarly transformed way*" (174). Vieta's species do not express a determinate amount, just like Diophantus' *eide*. The decisive difference is that the latter are just *provisionally* indeterminate, since they only specify the kind of the number sought until the actual determination at the end of the calculation. In Vieta's case, this determination is not mandatory. On the contrary, his *logistice speciosa* exclusively focuses on "general amounts" denoted by letter signs. What does it mean and how is it possible? The linear anachronistic explanation is substituted by a "circular" one such that it involves a "back and forth" movement.

On the one hand, to the "'general amount' in its indeterminateness, that is, its *merely possible* determinateness, is accorded a certain independence which permits it to be the bearer of 'calculational' operations" (174). This "independence" is a result of the "automatic" character given by manifest *rules of combination*. The behavior of the letter signs is entirely determined by them, so that *letters themselves* become *objects of calculation* (instead of actual things). In this way, "*the letter sign itself is literally perceived as something that exists independently of that which it functions to designate as a sign*" and, consequently, "*it functions as a symbol*" (Hopkins, 2011: 287). *Species* are from now on also *things* (i.e. items available for calculation) and as such

48 Hopkins clarifies:

> Klein holds that unlike the case in Diophantus' calculation with the species of number, where it is known in advance of the solution of any equation that the genus of the units involved in the solution will be homogeneous monads, in the case of the solution of Vieta's equations this has to be stressed, because the very indeterminacy of their solution does not render it all apparent that the ultimate referent of the species involved in each solution is, as Vieta believes, number in the traditional sense of *arithmos*.
>
> (2011: 291–292, n. 153)

are treated as numbers.[49] Letter signs gain, as it were, an independent and automatic life – they become "spiritual automata," to say it in Leibnizian jargon. But this is possible only on the basis of the stipulation accorded by the manipulatory rules that allow the substitution of letters with other letters in formulae and equations. At the end,

> it is obviously impossible to see "numbers" in the isolated letter signs "A" or "B," except through the syntactical rules which Vieta states in the fourth chapter of the *Isagoge*, contrasting them with the operational rules of the "*logistice numerosa*." These rules therefore represent the first modern *axiom system*; they create the systematic context which originally "defines" the objects to which they apply.
> (176)

Now, what is the origin of these rules? They explicitly come from ordinary calculations with determinate assemblages of things, as Vieta explicitly states with the law of homogeneity. They are now applied to *eide* and ciphers, despite originally governing counting with things, like fingers and *calculi*. "2a + 2a = 4a" is true *because* two units and two units make four units. Rules are thus the *link* between the practice of counting with determinate things as it is performed in pre-symbolic experience, where things are only given in "intuition," and the practice of calculating exclusively with species in algebraic thought. The difference is that even though the species ultimately behave like *arithmoi*, they are nevertheless "indeterminate." From the acquired level of indeterminacy, in fact, counting with determinate numbers appears as dealing with determinate cases, mere instantiations of the general formula. Therefore, what originally was an *arithmos* is now read as a particular "value" (a cipher). That is to say, a *symbol*. Diophantine expressions are read *as if* they were determinate equations pertaining to the *logistice numerosa*, to an algebra which differs only with respect to generality. In this way, Vieta sets the conditions to the *reversal* of pre-symbolic and symbolic experience and the complete concealment of the difference between the two forms of calculation (with *arithmoi* and *species*). *Arithmoi* are finally thought as *Zahlen*.

> Vieta sees in literal-reckoning only a more convenient, because more general, path to the solution of the problems posed. He can do this because he interprets the numbers with which Diophantus dealt from a higher conceptual level, because, in other words, he identifies the *concept* of number with the number itself; in short, he understands *Anzahl* as *Zahl*.
> (*LE*, 26)

Let us sum up the two moments just underlined. Even though they are *not* chronologically distinct, they can be distinguished. The dynamics includes the two poles: pre-symbolic counting with things and symbolic calculation. The relationship between what is original and what is derivative relies upon a *linking overturning* of the two poles – the link being the rules that anchor the meaning of symbols to the ordinary rules of counting, the overturning being the interpretation of the objects of

49 The sense-perceptible mark indistinguishable from the concept "is treated like other sense-perceptible things, for instance rocks, tables, copies of Klein's *Math Book*, and so on. Thus, in the case of the symbol '2,' the concept of twoness is at the same time understood as referring to two entities" (Hopkins, 2012: 292).

ordinary calculations as the same type of objects of the *logistice speciosa*. Vieta can give a symbolic independence to letter signs because he takes his rules from ordinary counting, but he can do it because he already has a symbolic understanding of ciphers. The original direction of the norms of the literal calculus from the operations with amounts of things is subverted and seen upside down. In the following passage, not only Klein explicitly characterizes the two moments just analyzed as two sides of the same coin, both inseparable conditions of possibility of algebraic thought, but he also relates them to the very fulcrum of his investigation. Since this circular movement is what is behind the constitution of symbolic entities, the unresolved tensions of modernity expressed, for example, by the "insurmountable difficulties" due to mathematical physics, have their very origin in this unnoticed ambiguity.

> And yet – *and here lies the germ of future difficulties* [my emphasis] – these rules are directly derived from "calculations" with numbers of monads. This means that a species can ultimately retain a numerical character even in its transformed mode and hence is able to become a "number," namely an object of "calculational" operations, only *because the ancient "numbered assemblages of monads" are themselves interpreted as "numbers"* [*Zahlen*], which means that they are conceived from the point of view of their symbolic representation.
>
> (*MB*, 176)

Even though Vieta maintains the ambiguity we exposed, preserving an original understanding of the concept of number, from which he takes the rules that govern his calculus with species, from now on, the symbolic interpretation of the rules and its objects will be the only one.

9.4.3 Vieta's error

The circular movement just explained exhibits a specific kind of "error." What Klein genealogically shows to be two *distinct* concepts of number, the Greek and the symbolic ones are thought as if they were the *same* in Vieta's own understanding. The latter collapses into the first one, and the difference between the two is therefore covered up. Being a limit-figure, Vieta represents the threshold between two conceptualities and shares some aspects of the two. His ambiguity resides in this "epochal" position which consciously preserves and unconsciously transforms an inherited understanding. In his case, the error is very specific, and it consists in identifying, ontologically, his own general quantities with Diophantus' *arithmoi*, and, linguistically, his symbols with Diophantus' signs. For example, "'2' is identified with the sign employed by Diophantus – and *this* is the decisive thing. The concept of twoness is at the same time understood as referring to two entities" (*LE*, 25). The identification of a reified concept *methodically* defined (e.g. "2") with any possible determinate group of two things is, as we pointed out, the condition of possibility for the constitution of his symbolic language. Without this exchange, no numerical significance could have been retained by letter signs.[50]

50 The same works in the case of geometry:

> All the curves investigated [...] are now on [starting from Descartes] nothing but symbolic exhibition of various possible relations, or of different 'functional' relations, between two (or more) variables. All this, however, is only one side of the matter [...]. It is no less essential that these *symbolic* curves were *understood* as the images of the curves exhibited by the Ancients.
>
> (*LE*, 19)

We said "error." But what does it mean? We do not mean something wrong *simpliciter*. This would be a "mistake," an exchange of what is true with the false – for instance, if I say "that is red," whereas it is in fact yellow. We do *not* argue that, for Klein, modern symbolic mathematics is founded on a mistake. It could appear so only if we take the non-symbolic conceptuality as the originally legitimate one: for the Greeks, a number is a certain amount of things and therefore it cannot be identified with a mark. However, the critical attitude that guides Klein's analysis prevents him from giving a definitive priority to one or the other conceptuality. It is not a matter of value or less value of a certain epistemological frame with respect to a given reality. The question is rather genealogical. We started from the fact of some "insurmountable difficulties" that surround modern science and such that motivated the inquiry. Now, what Klein disclosed, that is, the "error" responsible of the constitution of symbolic language, lies at the roots of these difficulties. This means that the fatal identification of sign and thing is not something that could be emended. Being *constitutive* of algebraic thought, it opens up a new scientific horizon. Namely, Vieta could create his new formalized objects *exactly because* he "wrongly" identified his "general numbers" with Diophantus' *arithmoi*. Since he thought he was dealing with the same thing, he invented a new one. It is this identification that leads to the retention of a numerical significance, which would be otherwise nonsensical. In this sense, it is not a "mistake," a misstep in the history of scientific thought, but rather the condition of possibility of a transformation which introduces science to a new world by occluding at once its difference with respect to the pre-symbolic one. This occlusion is *internal* to the possibility of the logic of symbolic mathematics.

Klein restates precisely this state of affairs by the employment of a Scholastic distinction. We can characterize the modern concept of number (*Zahl*) *prima facie* as second-intentional, that is, as having as object another concept, e.g., the general concepts "twoness" or "natural number." These are "general" insofar as they are not directly related to a certain amount of concrete or intellectual items. Therefore, their being is not *in re* but is exclusively dependent on the intellect, since the "concept" of a certain number is separated from the actual quantity of that number. Discussing Descartes' symbolic abstraction, Klein says that "the 'pure intellect', which, *being bare of any immediate reference to the world*, comprehends 'fiveness' as 'something separated' from 'five' counted point or other arbitrary objects – as mere 'multiplicity in general', as 'naked' multiplicity'" (*MB*, 201–202). Therefore, "'mere manyness' (*sola multitudo*), multitudinousness as such, which has its 'being' by grace of the 'pure intellect' is truly an *ens abstractum* or *ens rationis* in the sense of a '*second intention*'" (207).

However, thanks to the reification of it through the letter sign, it becomes at the same time the object of an *intentio prima*, allowing us to treat it, as we said, just like any other (countable) thing. You can think symbolically; you can, for example, add "a" to itself, because you think of the mark, which is a meaningless letter in itself, as an authentic number, without seeing differences of any kind – besides "generality" – with respect to determinate numbers.

> So, at first glance, "*a*" belongs [...] to the class of concepts which are applied not to individual objects but to concepts themselves. This, however, is only the first step. Actually, we deal with "*a*" and with all such algebraic numbers in exactly the same way as arithmetic deals with ordinary numbers. In other words, in algebra we use concepts of the second class as though they were concepts of the first

> class. […]. What we call a symbol is nothing else but a concept of the second class interpreted as a concept of the first class.
>
> (*LE*, 63)

The identification of *Zahlen* and *arithmoi* is thus accomplished by the reification of "general quantities," that is, by the identification of objects of second intentions with the objects of first intentions. This is the error that opens up the scientific project of modernity, and the origin of its perverted effects. According to the analysis we set forth in the second part, Klein cannot attribute to the Greek concept of number these paradoxical properties.[51] Being non-conceptual, an *arithmos* is the object of a first intention and therefore a first-intentional object. Thus, in general, the conceptuality of number in pre-symbolic thinking is first-intentional, whereas in modern thought it consists of the identification of the object of second intentions with the objects of first intentions.

9.4.4 A Cantorian example

The duplicity of the paradoxical transcendental structure of the modern concept of number according to Klein is supposed to guide every modern mathematical theory, and it should be detected *a fortiori* in the foundational one, i.e. set theory.[52] Take, for instance, Georg Cantor's first generating principle of transfinite numbers outlined in the famous 1883 article "*Grundlagen* einer allgemeinen Mannigfaltigkeitslehre":

> The sequence (I) of the positive real whole numbers
>
> 1, 2, 3, …, n, …
>
> has the basis of this generation in the repeated positing and uniting of basic units which are regarded as equal; the number n is the expression both for a definite finite number of such consecutive positings, as well as for the unification of the posited units into a whole. The formation of the finite whole real numbers thus rests on the principle of the addition of a unit to an existing, already formed number.
>
> (1976: 87)

The principle states the definition of the set of natural numbers and the conditions of their existence. This definition, together with the second generating principle, gives

51 "The identification between second and first intentional objects is 'unintelligible' because for 'natural' predication, to say that a concept is both general and particular 'at the same time' is nonsensical" (Hopkins, 2012: 295). In this sense, algebraic thinking represents a departure from "linguistic" *logos*, i.e. the *logos* fostered within, and from the regimentation of natural language, whose Aristotelian logic represents the paradigm. Despite the different account concerning the being of the symbol, Serfati (1999) agrees with Klein that it entails a *contradictory* "dialectic" unsolvable in natural language (in the "rhetoric register," as he says). Similarly, Cosgrove observes that "a second intention interpreted as a first intention would seem an example of what Husserl in *Logical Investigations* terms an 'impossible meaning'" (2008a: 231).

52 We follow here a laconic Kleinean suggestion given in 1932 (*LE*, 25).

the theorical justification for the introduction of transfinite numbers.[53] Now, even though the first part of the definition sounds Aristotelian – since Cantor speaks of "basic units regarded as equal" – a distinct symbol-oriented idea silently penetrates it. This is plainly shown, first of all, by the fact that the series of natural numbers begins from "1," which is not, for the Greeks, a number. He can conceive of a "basic unit" as a number just through the medium of the mark, so that the difference between the two (a basic item and "1") is eventually covered over. A unit can be a number because it is nothing but the symbol "1." It not only expresses, but it *is* the first natural number. Adding a unity to the first one, as the principle dictates, we obtain the number "2." While for the Greeks, as we said, the unity of the multiplicity composed of two units requires the hypothesis of an *eidos*, whose ontological status is different with respect to the monads it unifies; following Cantor's principle, on the contrary, the problem is not even considered. The unity of the number is expressed, as a matter of course, by the mark ("the number *n* is the expression both for a definite finite number of such consecutive positings, as well as for the unification of the posited units into a whole"). Therefore, "2" is "one" in the same sense as "1." The eidetic unity, in other words, is translated into a *symbolic unity*, i.e. a unity guaranteed by the determinate *mark* whose meaning is assured by the law of succession. Numbers are reciprocally *formally* distinguishable in virtue of the law of succession "n + 1" which is, in turn, considered by Cantor as equivalent to the ordinary operation of adding unities to other unities. The original one-many dilemma is thus put on a completely different conceptual level. As a consequence, even a transfinite number, that is, a number composed of an infinite number of unities, can gain its proper *symbolic* unity and become a new object – in Cantor's vocabulary, a *set*. A mark ("ω") and the syntactic rules determining its behavior are all that is needed to give to transfinite numbers a (symbolic) existence, no differently than in the case of irrational numbers, as Cantor himself notes.[54] The fact that he tries to justify this new type of number theologically, that is to say, by means of the hypothesis of God's capacity to collect an infinite multiplicity into one new object (a capacity precluded to human beings), demonstrates that he nevertheless wrongly identifies his symbolic numbers with the *arithmoi* of Greek mathematics. Because of this, he needs a metaphysical explanation of this mysterious unity (already accomplished by the symbol) that is *external* to the formal principle. Ancient Greeks did not face this problem for the simple fact that you cannot *say* how many units compose an "infinite" number. The enumeration cannot come to an end and thus any eidetic unity is in principle ruled out. On the contrary, Cantor can make a demand for this unity because for him the problem emerges *post festum*, i.e. after

53 The second generating principle says:

> if any definite succession of defined whole real numbers is given of which there is no greatest, then on the basis of this second principle of generation a new number is created which can be thought of as a *limit* of those numbers, i.e. can be defined as the next greater number of all of them.
>
> (Cantor, 1976: 87)

54 "The transfinite numbers are in a certain sense themselves new irrationalities and in fact in my opinion the best method of defining the finite irrational numbers is wholly similar to, and I might even say in principle the same as, my method described above of introducing transfinite numbers. One can say unconditionally: the transfinite numbers stand or fall with the finite irrational numbers; they are like each other in their innermost being; for the former like the latter are definite forms or modifications of the actual infinite" (Cantor, 1887: 99).

having already written down and manipulated certain symbols whose legitimacy is proved, as it were, by the symbolic practice itself. If he had not made this erroneous identification, though, how could he have seen a *number* in the symbol "ω"? As an heir of Vieta's transformation, he sees the relationship between non-symbolic numbers and its symbolic representation upside down and is incapable to notice the symbolic origin of the "magical" unity of infinite multiplicities. God's *ad hoc* intervention cannot account for it[55]: Klein's genealogy shows the reason why it is the case and suggests that its origin is more likely dependent on the logic of symbolic language itself.

9.4.5 Two misinterpretations of symbolic conceptuality

The Cantorian inclination to give an external justification for the use of mathematical symbolism is widespread. It is arguably due to the fact that signs transformed into symbols become the fetishistic interest of mathematicians and the actors of an open process that progressively extends the autonomy of symbolic methods with respect to the intuitions of ordinary experience. As the modern history of formal sciences demonstrates, the alleged "discoveries" of initially dubious entities (like complex numbers) or theories (like non-Euclidean geometries) are carried out by more and more extensive use of symbolic instruments. Once a certain symbol becomes indispensable for a theory, the "ontological exploration" (like Cantor's) into the rarefied mathematical continent takes place. If Klein correctly detects the origins of mathematical symbolism, then these concerns are derivative and, in the end, misleading. This kind of objective justification for the legitimacy of a certain calculus is dogmatic inasmuch as it misinterprets the logic of the formal language itself: philosophers exchange the reification of concepts in the medium of the letter as an actual "metaphysical" existence. They consider symbols only as objects of first intentions. Symbols are thus wrongly interpreted as signs, i.e. as having an objective denotatum.[56] And since the prejudice that natural numbers exist as a matter of course is dogmatically retained, it gives rise to questions like: "do variable numbers *actually* exist?" which can be rephrased in this way: "do variable numbers exist just like natural numbers do?"[57] The Kleinean perspective rejects these dogmatic problems since it shows that even "natural numbers," under Vieta's hands, have the ambiguous nature shared by any other

55 As Hallett notes: "Indeed, it seems that God is brought in essentially to bridge a gap (between a collection and its unity as a set) that we ourselves cannot bridge" (1984: 37).

56 Quite in accord with this diagnosis, Stenlund argues that modern thinkers still have not abandoned "the tendency to give meaning and significance to basic notions in mathematics and formal logic by translation or paraphrase into verbal language" (2015: 37). But, then,

> articulating *modern* mathematics in ordinary verbal language by assigning a place for mathematical propositions in the general category of propositions expressed by declarative sentences of natural language, have often resulted […] in mythological ways of thinking such for instance the 'non-linguistic' *Thoughts* of Frege (which have their special *Wirklichkeit* as entities of the 'Third Realm'), or the ontological mythologies of transfinite set theory of Cantor and Gödel.
>
> (2015: 35)

The heritage of an intuitive understanding settled in natural language is judged by him as the decisive flaw that prevents access to the correct interpretation of purely formal calculi, whereas for Klein the need of translation, or rather the grounding of symbolism in pre-symbolic experience, is something that cannot be severed once and for all.

57 I refer to the problem tackled by Frege in exactly these terms in *What is a function?* (1904).

symbol. Arabic ciphers are symbols, although they appear to be closer to the groups of objects given in "intuitive" experience.[58] Through Vieta's transformation, they are submitted to the peculiar dialectic of generalization and reification we explained above. The gap between mathematical symbols and the sensible world is thus irreversible and cannot be restored, or "redeemed," by means of the mediation of a realm of objects allegedly grounded in it. Therefore, the logicistical attempt to define all kinds of numbers in terms of natural numbers (and eventually in terms of sets) does not represent a valid solution for the justification of the existence of any mathematical entity. Since all numbers lie on the same symbolic ground, their "existence," in Vieta's algebra and in Descartes' geometry as well, does not rely upon any commitment to entities in the "real world." On the contrary, because equations are now formulas for the *production* of determinate geometric figures or arithmetical solutions, to exist as an individual, to be "actual," simply means to *be* the value of a variable.[59] Of course, Quine's famous motto is echoed in these words.[60] This shows both the original debt that modern logic pays to symbolic mathematics and, as a consequence, the reason why its correlated ontology is an insufficient ground for the latter – simply because it embodies and thus reiterates the very same "logic" of symbolic mathematics itself.[61]

Analogously, the opposite philosophical doctrine which thinks of mathematics as an empty game of signs, thus reducing its content to the mechanical manipulation of them, is equally unilateral and misses a crucial aspect as well. In fact, Klein does *not* say that the original meaning of *arithmos*, in the case of algebra, is completely *lost*,

58 "In quanto simbolo, il 'numero naturale positivo' dà solo l'illusione di essere una quantità concreta riferita alle cose del mondo, quando in realtà è già un costrutto simbolico esattamente come gli interi negativi, gli irrazionali e gli immaginari. La matematica simbolica è così fondata sulla continua illusione di star parlando di oggetti quando invece ha di mira solo costrutti segnici che ottengono un significato perché sottoposti alle regole di calcolo" (As a symbol, the 'positive natural number' only gives the illusion of being a concrete quantity referring to the things of the world, when in reality it is already a symbolic construct exactly like the negative integers, the irrational and the imaginary. Symbolic mathematics is thus founded on the continuous illusion of talking about objects when instead it only targets sign constructs that obtain meaning because they are subject to the rules of calculation.) (Chiaravalli, 2018: 32).

59 Talking about Descartes' geometry, Cobb-Stevens remarks:

> L'équation n'expose pas les caractéristiques d'une figure intuitionnée ou intuitionnable; bien plutôt, elle présente une formule de la production des figures. Chaque évaluation des variable liées au sein de l'équation produit une section conique déterminée à partir de ce que D. Lachterman appelle 'un continuum de possibilités abstraites.' Bref, pour Descartes, être *modo geometrico*, c'est être la valeur d'une variable.
>
> (The equation does not expose the characteristics of an intuited or intuitable figure; rather, it presents a formula for the production of figures. Each evaluation of the linked variables within the equation produces a conic section determined from what D. Lachterman calls 'a continuum of abstract possibilities.' In short, for Descartes, to be *modo geometrico* is to be the value of a variable.)
>
> (1997: 101–102)

60 Cobb-Stevens (1997), Cosgrove (2008a) and Serfati (1999) have underlined the symbolic roots of Quine's ontology (inherited through Frege's logic).

61 Klein explicitly makes this connection in the *Math Book*:

> In Vieta's 'general analytic' this symbolic concept of 'number' appears for the first time, namely in the form of the *species*. It lies at the origin of that direct route which leads, *via* the 'characteristica universalis' of Leibniz, straight to modern theories of 'logistic' (i.e., that branch of symbolic logic dealing with the foundation of mathematics). The condition for this whole development is the transformation of the ancient concept of *arithmos* and its transfer into a new conceptual dimension.
>
> (185)

but that it is *preserved*, albeit in a peculiarly transformed way. Signs do *not* become meaningless items, but *symbols*. Since they are subjected to the rules stemming from ordinary calculation, they retain a *numerical* character anyway. Therefore, the reference to the pre-symbolic world, even though occluded, cannot be denied. Formalism, instead, tries to detach itself from any "intuitive" or "imaginatively" experience – mainly from the fetters of natural language. The concepts it represents do not have any fixed extant denotatum, but rather are exhaustedly defined by their mutual relations established by the axioms. In other words, they are considered only as objects of second intentions neglecting their original connection with the pre-symbolic world.

In his wonderful study on the formalization of logic, Lachterman (1987) criticizes the formalistic ambition to uproot its concepts from any "intended," that is, pre-formal and spurious interpretation. One of the dreams of the modern *mathesis universalis*, he claims, is to "paint the thought," that is, to expose pure thinking in the concrete and intuitable medium of an ideal language, where thereby signs could "think for themselves."[62] However, he argues that the detachment of a pure objective system of signs from the meanings originally intended is shown to be chimerical by the very formalistic proofs that lead to the limitation theorems like Gödel's.[63] The uprooting of formal thinking from the life-world is doomed to failure. The origins of this failure trace back to the forgetful transformation of the concept of number first realized by Vieta in the *Isagoge*.

9.4.6 Some concluding remarks

What can we learn from this final theoretical *tour de force*? We have outlined the scene on which Klein's ideas are staged in the first part. Through the phenomenological study of the transformation that occurred to the concept of number, Klein traced the difference between the two conceptualities hypothetically assumed in advance. Given the importance of mathematics in the constitution of scientific thought, both in ancient and modern times, Klein focuses on the paradigmatic nature of the concept of number, and then generalizes his results. Greek conceptuality is first-intentional, in the sense that it builds its concepts by abstraction from immediate experience, whereas symbolic conceptuality consists of the identification of the objects of second intentions with those of first intentions, that is, the objects represented with the means of their representation.

This result justifies our characterization of this difference as incommensurable, in the twofold sense that the two conceptualities do not have a common measure and that their respective languages cannot be equivalent for structural reasons. From the standpoint of Greek conceptuality and natural language, symbolic concepts appear as paradoxical, if not absurd. From the standpoint of modern conceptuality and symbolic languages, the concepts of Greek philosophy and mathematics appear as less refined and still affected by the vicious influences of sensibility.

62 A striking passage that testifies to this idea comes from Heinrich Hertz:

> We cannot escape the felling that these mathematical formulas have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than was originally put into them.
>
> (quoted in Stenlund, 2015: 21)

63 This is his general conclusion: "Far from being an escape from the insecurities occasioned by pre-formal mathematical logic [...], formal metamathematics remains wedded to informal discursive practices" (1987: 217).

Incommensurability, however, does not amount to discontinuity – a concept that rather pertains to the anachronistic accounts. We argued that, despite the appearances, Klein's account is, at the end, based on continuity. This assumption is confirmed by the fact that Klein's interpretation of Vieta's transformation shows that it does not merely consist in the substitution of one conceptual paradigm for another one, which is supposed to explain better an already given domain of objects and laws. The system is dynamic or non-linear because the previous system is absorbed and transcendentally transformed – and not simply overridden. This transformation, in turn, is possible since it was unperceived: it is only because the symbolic method is thought as a mere *renovation* of a technique already hiddenly employed by the Greeks, that it could be established as an *innovation*. The new method is symbolic only insofar as its objects are erroneously identified with the non-symbolic ones. Thus, the continuity is not determined by an objective permanence of objects (as radical anachronists believe) but rather by the subjective way of assimilation and transformation of the objects themselves – again, "subjective" here is not to be confused with "psychological." Vieta is the proportional mean, as it were, between two conceptualities since he takes for granted the Greek one, but at the same time, he betrays it because of his epochal error of identification objects of first and second intentions.

This is the reason why, secondly, this difference has been occluded and the origin of modern conceptuality forgotten from the beginning. Just like the interpretation of the historical development, a complete *bouleversement* occurs with respect to the original relationship between reality and language. Instead of starting from actual multiplicities offered by sensible or intellectual experience, and then expressing them in natural language, from now on "to think" will mean – first of all – "to think symbolically," i.e. operating efficiently with language or calculating with ciphers. The real world is no longer epistemically expressed by natural language; it is rather paradoxically identified with symbolic language and its rules, just like *arithmoi* are eventually identified with ciphers.

Finally, Klein's investigation can be regarded as transcendental because it does not take into consideration the nature of a certain object *in abstracto* already given in sensorial or intellectual experience, but rather suspends its validity and its being given so-and-so, in order to look back into the origin of this givenness. The dogmatic concerns about the "real" nature of this or that object are left aside, and the subject discussed regards the way in which that very object is experienced and formed, along with the way in which this experience changes over time. With respect to numbers, the examination shows that an actual transformation of this modality took place starting from the sixteenth century – a transformation that arguably has impacted every other scientific field.

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10 Jacob Klein on François Viète and the birth of the modern symbolic concept of “number”

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Abstract: Jacob Klein’s philosophical-mathematical account of the origin of the basic concept in modern mathematics—the symbolic concept of number—is explicated, paying close attention to the distinction between the immanent mathematical and philosophical aspects of his account. Crucial to the former is the generalization of the being and concept of number in Viète’s Analytic Art and the formalization of the object of mathematical analysis, while crucial to the latter is the symbolic mode of being proper to both the presentation and concept of the generalized and formalized object of analysis.
Keywords: Jacob Klein; François Viète; Diophantus, Number; Concept of Number; Mathematical Analysis; Algebra.

10.1 Introduction

In a letter to Leo Strauss, dated April 17, 1937, Jacob Klein characterizes his then unpublished manuscript (bearing at that time the title) “Greek Logistic and the Origins of the Language of Algebraic Formulae” as follows: “So far as I am competent to judge, I can only say: this work is the first attempt to develop a *fundamentally* different approach to the history of the exact sciences *and* philosophy as it is ordinarily practiced.”[1] He goes on to say: “Probably no one will notice this point, just as it’s likely no one will point out the actual, factual [*Sachlich*] weaknesses in this work.” My discussion of Klein’s account of the birth of the modern symbolic concept of number in François Viète’s *Introduction to the Analytical Art*[2] will attempt to take notice of this first point, by fixing as its point of departure precisely what is fundamentally different in Klein’s approach to the history of the exact sciences and philosophy. Clarity on this point is crucial, in my view, not only for understanding his account of the

1 In Leo Strauss, *Gesammelte Schriften*, ed. Henrich Meier (Stuttgart: Metzler, 2001), III: 455–605, here 499.

2 Francisci Vietae, *In Artem Analyticem* (sic) *Isagoge*, Seorsim excussa ab opere restituate Mathematicae Analyseo, seu, Algebra Nova (*Introduction to the Analytical Art*, excerpted as a separate piece from the *opus* of the restored Mathematical Analysis, or *The New Algebra* [Tours, 1591]). English trans. J. Winfree Smith, *Introduction to the Analytic Art*, appendix to Jacob Klein, *Greek Mathematical Thought and the Origin of Algebra*, trans. Eva Brann (Cambridge, MA: The M.I.T Press, 1968). Hereafter cited as “*Analtyic Art.*”

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birth of the symbolic concept of number in Viète's thought but also for assessing any errors that it may contain.

On Klein's view, both the history of the exact sciences and philosophy as usually practiced presuppose the historical invariance of what he terms the "conceptuality" (*Begrifflichkeit*) behind the formation of exact scientific—i.e. mathematical—and philosophical concepts. For instance, historical studies of mathematics typically suppose that the mathematical objects under study, notwithstanding their being historically dated, are accessible in their "true being" from the conceptual level of contemporary mathematics. This is the case for Klein in both the extreme and moderate cases where this supposition is in effect. In the extreme case, for instance, the mathematical objects operative in Euclidian geometry are supposed to be mathematically equivalent with their symbolically formalized counterparts in contemporary mathematics. In the more moderate case, the historical developments in mathematics are supposed to be inevitable stages on the way to the mathematics of the present. Likewise, in philosophy as usually practiced, according to Klein, the supposition is operative that its historically dated concepts are both accessible and assessable in accordance with a common ahistorical conceptuality. In the extreme case, for instance, it is supposed that the basic concepts of Greek ontology, the εἴδη, are accessible and assessable in terms of the modern understanding of "ideas" as mental contents, as "mental presentations" (*Vorstellungen*). Or, more moderately, continuity in the development of the basic concepts of philosophy, their historically datable character notwithstanding, is un-problematically supposed.

10.2 Klein's identification and response to the presuppositions behind contemporary history and philosophy of mathematics

The fulcrum of the fundamental difference in Klein's approach to the history of exact sciences and philosophy from these modes of approach concerns the lack of *philosophical* warrant he maintains is behind the supposition of the historically invariant conceptuality of mathematics and philosophy. For Klein, this lack of warrant is manifest above all with respect to the presentation of both the history of the origin of modern symbolic mathematics and the philosophical significance of the conceptual structure of its formal language. Regarding this history, its conventional presentation "always takes for granted, and far too much as a matter of course, the *fact* [*Faktum*] of symbolic mathematics."[3] And, "Moreover, most of the standard histories attempt to grasp Greek mathematics itself with the aid of modern symbolism, as if the latter were another external 'form' which may be tailored to any desirable 'content'" (*GMTOA*, 20/5). The problem here, on Klein's view, is twofold. On the one hand, the "*character* of the conceptual transformation" in mathematics from pre-modern, and especially Greek, mathematics, to modern symbolic mathematics, is not patent in the conventional presentations of this history. On the other hand, the lack of a "sufficient

3 Jacob Klein, "Die griechische Logistik und die Entstehung der Algebra," *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik,* Abteilung B: *Studien*, vol. 3, no. 1 (1934), 18–105 (Part I), and no. 2 (1936), 122–235 (Part II), here 20; English translation: *Greek Mathematical Thought and the Origin of Algebra*, trans. Eva Brann (Cambridge, MA.: MIT Press, 1969; reprint: New York: Dover, 1992), 5. Hereafter cited as "*GMTOA*," with German and then English tr page numbers.

account (*genügende Rechenschaft*)" of this "*character*" is shared by philosophy, when its task is to grasp the conceptual structure of modern symbolic mathematics. The result of this, philosophically speaking, is that the understanding of this structure is determined "from a conceptual level which is, from the very beginning, determined by modern [symbolic] modes of thought."

To "disengage… as far as possible from these [modern] modes of thought" is therefore the "first concern" of Klein's fundamentally different approach to the history of exact science and philosophy. According to him, this disengagement is necessary for a historically and philosophically unbiased investigation into the origin and conceptual structure of symbolic mathematics' formal language. That is, the historical and philosophical priority of modern symbolic modes of thought cannot be supposed as a matter of course, if the goal of inquiry is to secure access to that origin and conceptual structure in a manner which, from the start, presupposes neither the historical knowledge of the sources of that origin nor the philosophical knowledge of the conceptual structure of symbolic mathematics whose origin is in question. The attempt to dispense with this priority is what is behind Klein's talk of mathematically and indeed philosophically *distinctive* conceptualities governing the concepts operative in ancient and modern exact sciences. Thus, on the one hand, there is the ancient conceptuality, characterized above all by the inseparable relationship of its concepts to a multitude of beings *other* than those concepts. The paradigmatic example of this for Klein is the Greek concept of ἀριθμός, whose conceptuality as an "ideal" being is characterized as the determinate unity of a multitude of determinate beings. On the other hand, there is the modern conceptuality, characterized above all by the primary relationship of its concepts to other concepts, such that

> Nothing but the internal connection of all the concepts, their mutual relatedness, their subordination to the total edifice of science, determines for each of them a *univocal* sense and makes accessible to the understanding their only relevant, specifically scientific, content.
>
> (*GMTOA*, 125/121)

The paradigmatic example of this for Klein is the modern symbolic concept of number (*Zahl*), whose conceptuality as an "ideal" being is characterized by its relation to other concepts, above all, to the symbolic calculi that it underlies, in a manner that leaves completely indeterminate its relation to beings *other* than concepts. Thus, above all for Klein, the symbolic concept of number is related in a completely indeterminate manner to the determinate beings whose multitude the "ideal" being of the Greek concept of number as ἀριθμός was characterized as uniting.

10.3 Klein's identification of the creation of the formal language of mathematics with the foundation of modern algebra

Klein's interest in the origin and conceptual structure of the formal language of symbolic mathematics is driven by his conviction that there is an "intimate connection of the formal mathematical language with the content of mathematical physics" (19/4), such that "it is impossible, and has always been impossible, to grasp the meaning of what we nowadays call physics independently of its mathematical form." This state of affairs informs Klein's view of the priority of the investigation of the origin and conceptual structure of the formal language of modern symbolic mathematics over other fundamental questions "concerning the inner relation between mathematics and physics, of 'theory' and 'experiment', of 'systematic' and 'empirical' procedure within mathematical physics." Klein's stress on the importance of investigating the "origin" in question has its basis in his view that "The creation of the formal language of mathematics is identical with the foundation of modern algebra."

Klein's account of the identity of the "foundation" of modern algebra with its "historical" origin, in turn, stems from the following line of thought. One, exact sciences are in need of a "foundation," meaning an account that clarifies or otherwise renders perspicuous their basic concepts, an account that does so to the end of justifying these concepts' cognitive pretentions to provide exact scientific *truth*. This necessity is at once scientific and philosophical. It is scientific insofar as the sciences in question understand themselves to be in the service of truth. It is philosophical insofar as it is not a function of the sciences in question, for instance, modern symbolic mathematics, to justify their cognitive pretention to truth beyond the ambit of their formal structure *as defined by the science in question*. Two, contemporary philosophy is unable to provide the exact sciences their necessary foundation, as evidenced by its "struggle to fix the principles of mathematical physics" (155/152) as well as by the foundational "battle" (175/169) waged today whether "formal logic" or "mathematical logic" has primacy in representing or otherwise modeling the "true" structure of scientific cognition of the world. And, three, the foundational failure of contemporary philosophy with respect to the exact sciences is not an accident but has its basis in an unresolved tension in the science founded by the early moderns. On Klein's view, this is the case because the founders of modern science "were not, for the most part, themselves aware of their own conceptual presuppositions" (155/152). Most specifically, they did not realize that "the manner in which" they "set about attaining a mathematical comprehension of the world's structure betrays, from the outset, a different conception of the world, a different understanding of the world's being," then the ancients'. The result of this, according to Klein, is a tension in the science they founded, between the legacy of the ancient non-formalized conception of the knowledge of the true world and the modern symbolically mathematical conception of that knowledge.

These last two points are what is behind Klein's historical mathematico-philosophical investigation of the "origin" of the formal language of mathematics, namely, the thesis guiding that account that the sources of that formal language's basic concepts, together with their cognitive justification, are coincident with "the foundation of modern algebra." As we can now see, the "foundation" in question here for Klein is not "historical" in the sense of the documentation of the *res gestae* connected with modern algebra's origin. Rather, it is "historical" above all in the sense of the reconstruction of the dated conceptualities—ancient and modern—involved in the transformation of mathematics at issue in the foundation of modern mathematics. The reconstruction of these conceptualities, in turn, has its basis for Klein in the factual (*Sachlich*) presentation of the concepts and cognitive relations belonging to the ancient and modern mathematics involved in this transformation.

10.4 Klein's historical identification of Viète's Analytical Art as the origin of the foundation of modern algebra

In broad overview, Klein's factual account of the foundation of modern algebra locates its origin in François Viète's Analytical Art, specifically in the mode of analysis he calls "*logistice speciosa*," and more particularly still in the kind of analysis he calls "zetetic." According to Klein, this origin has two hallmarks. One, the concept of number as an indeterminate magnitude, that is, as "general number." Two, a symbolic calculus, the syntactical rules for which create "the systematic context which originally 'defines' the object to which they apply" (184/176). Klein traces this origin to three distinguishable yet nevertheless interrelated sources. (1) The absorption in the West, from the thirteenth until midway through the sixteenth century, of "the Arabic science of 'algebra' (*al-g'abr wa'l-muqâbala*) in the form of a theory of equations, probably itself derived from Indian as well as from Greek sources" (19/4). (2) The "elaboration, particularly in Italy, of the theory of equations which the Arabs had passed on to the West" (20/5). And (3), the text of one of the Greek sources, the *Arithmetic* of Diophantus, which exerted a special influence on the content and form of the Arabic science, and which was rediscovered in the West in the fifteenth century. On Klein's view, this last source of the origin of modern algebra is the most crucial, as he credits Viète's decisive broadening and modifying of the logistical technique found therein with the foundation of modern mathematics.

Klein's account of the origin of the modern symbolic concept of number in Viète's zetetic analysis distinguishes between its immanent mathematical structure, Viète's interpretation of its historical significance, and the philosophical analysis and assessment of the conceptual structure of both the symbolic concept of number and Viète's historical interpretation of it. All three of these moments are parts of what Klein refers to as the "foundation" of modern algebra. However, only the first two moments are directly connected with Viète's understanding of his Analytical Art, as the third involves the philosophical, and indeed, unique philosophical, perspective brought to bear by Klein on the conceptual structure of that art. Examples of the foundational moment of immanent mathematical structure would be Viète's designations of unknown magnitudes as "ladder magnitudes" (whose "rungs" or "degrees" are Side or Root, Square, Cube… [Latus seu Radix, Quadratum, Cubus, …]) and known ones in terms of their genera (Length or Breath, Plane, Solid [Longitudo latidudove, Planum, Solidum], …), together with his presentation of the four "precepts" of "reckoning by species" and the "law of homogeneity [*lex homogenerorum*]." Examples of the foundational moment of historical interpretation would be Viète's presentation of a connection between his "law of homogeneity" and Adrastus, and his belief that Diophantus' arithmetical books "employed zetetics." Examples of the foundational moment belonging to the philosophical clarification of the conceptual structure of the concept of symbolic number would include the following claims of Klein. (1) This conceptual structure is the result of Viète's universalizing extension of Diophantus' concept of εἶδος (species) beyond the realm of numbers (understood as the determinate unity of multitudes of monads) while nevertheless retaining its tie to that realm, such that the species for Viète means "general number." (2) The philosophical structure of the conceptual transformation that generates the concept of symbolic number exhibits an object of a second intention assuming the status of the object of a first intention. (3) This transformation gives birth to the distinctively modern conceptuality made evident, above all, by its contrast with the ancient conceptuality that precedes it.

Despite these distinctions, Klein views these moments as interrelated, with the immanent mathematical moment functioning as the basis for its author's (Viète's) historical interpretative moment, and with both of these moments functioning as the basis for the moment of philosophical analysis and assessment that Klein himself provides of their conceptual structures. Cognizance of the interrelated structure of these distinctions is important for properly grasping Klein's account of Viète's foundational role in the birth of the modern symbolic concept of number. The failure to take such cognizance, for example, to take into account the basis the first two moments provides for the moment of philosophical analysis and assessment, can be seen to be behind Albrecht Heeffer's fundamentally mistaken claim that Klein's account of the emergence of symbolic algebra is "rooted in German idealism, where concepts realize themselves with the purpose to advance mathematics."[4] Or, more generally, this failure is behind the claim that Klein's account of the history of mathematics is ontologically over-determined. Of course, despite making these foundational distinctions, Klein's execution of his approach to the history of the exact sciences may make factual errors in its presentation of that history and indeed may present philosophically questionable accounts of the conceptual structure of the mathematics involved. However, in order to be cognitively responsible, criticism along these lines must attempt to establish itself on the basis of Klein's text.

10.5 Klein's mathematical account of the universalization of Diophantine "species" in Viète's Logistice Speciosa

Turning now to Klein's text, its presentation of the foundational moment of immanently mathematical structure in Viète's Analytical Art focuses on the "species." On Klein's account, the mode of analysis Viète designates "*logistice speciosa*" exhibits four salient characteristics. (1) The units upon which its calculation operates are capital letters of the alphabet, with no intrinsic signification, alphabetical or otherwise (e.g. mathematical). (2) The signification of these letters is stipulated by the Analytical Art's "laws," specifically those of zetetic analysis. The laws of zetetic analysis use capital letters to distinguish given (known) magnitudes from "undetermined unknowns," by designating known magnitudes with vowels and the unknown with consonants. These designations function to distinguish artfully these two kinds of magnitudes "by a constant, everlasting and very clear symbol" (340). Calculating with symbols is characterized as "reckoning by species (logistice speciosa)" (328), which "operates with the species or forms of things." (3) The "*canonical rules of species calculation*" (180/172) articulate the rules for adding, subtracting, multiplying, and dividing species, to which are joined the degree of the unknown (beginning with the second degree) or the genus of the known. (4) The "supreme and everlasting law of equations or proportions" (324), the "law of homogeneity," which holds that "Only homogeneous magnitudes are to be compared with one another."

According to Klein, Viète's species' "rung" or "degree" designations correspond to the εἴδη of numbers (ἀριθμοί) in Diophantus' *Arithmetic*, which are "independent of *each* multitude of monads (πλῆθος μονάδων) [composing a number (ἀριθμός)] and in this sense 'general' (καθόλου)" (149/145). Despite this independence, however, for Diophantus the meaning of the species is inseparable "in each case from a determinate number (*Anzahl*) of monads." However, when in Viète's equations letter signs

4 Albrecht Heeffer, "On the Curious Historical Coincidence of Algebra and Double-Entry Bookkeeping," *Foundations of the Formal Sciences VII Bringing together Philosophy and Sociology of Science*, eds. K. François, B. Löwe, T. Müller, B. Van Kerkhove, (London: College Publications, 2011), 111–132, here 115.

are either adjoined—beginning with the second degree—to the rung of the species, or, in the species of the first degree, when the rung is not appended to the letter sign and therefore coincides with it, Diophantus' concept of εἶδος is "universalized" in the following sense. In Diophantus, the meaning of the species, despite its analytic independence in calculation from a determinate number of monads, is nevertheless inseparable from determinate numbers, because the solution of these calculations is realized in a synthesis employing determinate numbers. This is not the case for Viète. Rather, for him, adjoining the letter sign to the species together with its outright identification with that sign shifts the meaning of the species to the indeterminate unknown magnitudes artfully designated by the letter sign's "constant, everlasting and very clear symbol." This shift in the species' meaning allows Viète's zetetic to resolve problems analytically, that is, in terms of its calculations with indeterminate magnitudes, thus dispensing with the Diophantine synthesis with determinate numbers. With this, the concept of number as "general number," that is, as an indeterminate quantity, comes into being.

10.6 Klein's mathematical account of Viète's formalization of the species in the service of pure algebra

In addition to this arithmetic universalization of the species, Klein maintains that Viète formalizes it, in the sense that both the known and the unknown magnitudes, which are artfully symbolized in the formulation of the equation or proportion "through which the magnitude sought is *itself* produced" (171/167), are "indifferently applicable to numbers and to geometrical magnitudes" (169/165). Thus, for Klein, Viète "devotes the 'logistice speciosa' to the service of '*pure*' algebra, understood as the most comprehensive possible 'analytic' art," that is, as a "*general* algebra which will be equally applicable to geometric magnitudes and numbers" (162/158). Klein finds the evidence for this in Viète's distinction between "zetetic" analysis, wherein equations or proportions are set up exclusively on the basis of species calculation, and "rhetic" and "exegetic" analyses, which, respectively, compute magnitudes arithmetically or construct them geometrically, depending on whether the required magnitude "is to be expressed in number" (346), or is to be shown in regard to "lengths, surfaces, and solids."

For Klein, it is significant that Viète did not interpret his Analytic Art as an innovation, as something new, but as the renovation of an art already in the possession of the ancients but which they chose to hide. Specifically, Viète maintained that Diophantus used "zetetic," although sufficiently subtly so that it appeared "*as though it were only founded on numbers and not also on species—although he himself used them*" (177/170). On the one hand, this is significant for Klein because it provides strong evidence that Viète himself understood the Analytical Art "generally," being founded on numbers no less than on species. On the other hand, Viète's "interpretation of the Diophantine *Arithmetic*, which views it exclusively as an 'artful' procedure, and has but small interest in the determinate results of the solutions" (178/170–171), "prescribed to historical research the approach which governs it to this very day": namely, "the matter-of-course acceptance within modern consciousness of the revolution in the ancient mode of forming concepts and interpreting the world, which first took shape when Viète founded his 'general analytic'."

Taken out of its proper context in Klein's text, this last and other similar seemingly bold philosophical statements would seem to lend credence to the criticism of ontological over-determination directed at his claim, that in Viète's concept of species "the modern concept of 'number' [*Zahl*] is born" (183/175). However, when reinserted back into that context, Klein's claim must be understood in terms of its relation to both Viète's interpretation of Diophantus' *Arithmetic* and the immanent mathematical structure of the paradigmatically ancient (and therefore pre-modern) concept of number in Greek mathematics. As we have seen, Viète attributes to Diophantus an understanding of species in his (Viète's) sense, that is, as symbols that artfully designate both unknown and known (given) magnitudes. As we have also seen, Viète reckons with these magnitudes in zetetic analysis in a manner that treats them as numerically and geometrically indeterminate. What makes Viète's interpretation of Diophantus so significant for Klein's *philosophical* analysis is not simply that, *mathematically* speaking, it is wrong, but also that the mistake behind it is repeated every time the assumption is made regarding pre-modern mathematics that the generality of its method entails—just as methodological generality entails for modern mathematics—the generality of its object. Klein draws two crucial interrelated philosophical implications from this state of affairs. One, so long as it is in effect, the true nature of the objects of pre-modern mathematics will remain inaccessible to thought, and necessarily so. Two, the nature of the conceptual transformation from pre-modern (beginning with the concepts of ancient Greek mathematics) mathematics to modern will likewise remain inaccessible to thought.

10.7 Klein's philosophical account of the transition from the pre-modern to modern concept of number in Viète's zetetic analysis

As we've seen, making this transition accessible to thought is precisely the task Klein sets for his mathematico-philosophical investigation of the foundation of modern symbolic mathematics, by seeking that foundation in its historical origin in Viète's Analytical Art. Klein does so by posing and answering the following questions. One, how does the "general number" that comes into being with the symbolic universalization of the Diophantine species in Viète's zetetic analysis—its quantitative indeterminateness notwithstanding—nevertheless come to be "accorded a certain independence which permits it to be the subject of 'calculational' operations" (182/174)? Two, how is it that Viète's symbolically universalized species "can ultimately retain a numerical character [*Anzahl-Charakter*] even in its transformed mode and hence is able to be a number [*Zahl*], namely an object of 'calculational' operations"?

Klein's answer to the first question is that Viète's species is a symbolic formation whose conceptual structure, and radically transformed dimensionality vis-à-vis that of pre-symbolic number, is rendered philosophically perspicuous by the mediaeval distinction between "first" and "second" intentions (and first and second intentional objects). According to Klein, on the one hand, a concept that immediately means a concept and not an "entity" (*Seiende*) may be characterized as a second intentional object and therefore as the object of a second intention. On the other hand, for Klein, a concept whose meaning refers directly to an entity, without the mediation of other concepts, may be characterized as a first intention and its object a first intentional object. This distinction permits Klein to characterize—*philosophically*—the pre-modern concept of number, that is, a determinate amount of determinate monads, as first intentional, because the articulated structure of the multitude of these monads presents a field of entities that are *other* than the conceptual unity brought to bear on it by the number itself or by its εἴδη. Klein does not claim, however, as it is often mistakenly asserted, that the modern symbolic concept of number is simply the object of a second intention. Rather, he is both explicit and consistent in maintaining that when Viète adjoins the letter sign designating the unknown magnitude to the Diophantine εἶδος, this

> *transforms the object of the* intention seconda, namely the 'general number' meant by the letter sign, *into the object of an* intention prima, of *a 'first intention'*, namely of an 'entity' which is directly apprehensible and whose counterpart in the realm of ordinary calculation is, for instance, 'two monads', 'three monads.'
>
> (182/175)

10.8 Klein's philosophical account of the symbolic "being" of the modern concept of number

Klein's philosophical account of the conceptual structure of the modern concept of "number" (*Zahl*) therefore characterizes that structure as the "symbolic conception and symbolic presentation [*symbolisch begriffen und symbolisch dargestellt*]" (183/175)—in the "medium of the species"—of the "general number" as "an in itself objective *formation* (*als in sich gegenständliches* Gebilde)." By "symbolic conception," he understands Viète's assignment of letters of the alphabet to known and unknown magnitudes, such that both (and not just the unknown, as in Diophantine *Arithmetic*) are rendered indeterminate. When the magnitudes in question are numbers, "the 'how many' (*Wieviel*) is provisionally left indeterminate" (169/165), an indeterminacy Klein maintains is coincident with the transformation of "number" from *Anzahl* to *Zahl*; that is, from a determinate amount of determinate *monads* to the "character" of being an amount that leaves the "amount's" determinate amount undetermined. In Klein's words:

> While every ἀριθμός means *immediately the things or the units themselves* whose "definite amount" it exhibits exactly, Viète's letter signs [first of all mean precisely this *concept* of an amount as a determination insolubly related to things or units, and thus they][5] *immediately mean the general character of being an amount* that belongs to every possible definite amount—i.e., "amount overall" [*Anzahl überhaupt*]—and only *mediately the things or units themselves* that may be present in any particular definite amount.
>
> (182/174)

By "symbolic presentation," Klein means the sense-perceptible letter sign, that is, an object given in perception—as a mark on the page or the blackboard—being apprehended *as* something other than a random letter of the alphabet, namely, *as* a number in the sense of "general number." This "symbolic" presentation, together with the distinct but nevertheless inseparable from its presentation symbolic concept of

5 The clause in brackets renders the German original, "meint zunächst einmal das Buchstaben-Zeichen bei Vieta eben diesen *Begriff* der Anzahl als einer auf Dinge bzw. Einheiten unablöslich bezogenen Bestimmung," which is not translated in the English translation.

the numerically general character of being an amount, characterizes, according to Klein, the symbolic "being" ["*Sein*"] (182/175) of Viète's species. This, the "being" of the objects of Viète's Analytical Art, and more precisely, of its zetetic mode of analysis, is on Klein's view only intelligible "within the language *of symbolic formalism*." This is the case, because as symbolic formations, the species are not intelligible in terms of their pre-modern conceptuality, which goes back to the ancient Greeks.

Thus, Viète's species can be apprehended neither as numerically independent Pythagorean and Platonic εἴδη (i.e. as the non-numerical unities of the Odd and Even, or the likewise non-numerical unities of Platonic "ideal" numbers), nor as Aristotelian "abstractions" (i.e. as "noetically" reduced sense-perceptible beings). The language of symbolic formalism is "fully enunciated first in Viète," which for Klein means that his zetetic analysis effectively made possible the mathematical *formula*, and with that, "above all, a new way of 'understanding,' inaccessible to ancient ἐπιστήμη, is opened up" (183/175). For Klein, working in the 1930s and therefore at a time when the foundational crisis in mathematics at the turn of the century and the early decades of the twentieth century was still fresh, the claim that the modern concept of "number" as *Zahl*, "as it underlies symbolic calculi, is itself, as is that which it means, *symbolic in nature*" (183/176), if not uncontroversial, would have at least been comprehensible along roughly the following lines. "Pure" mathematics, not being in the service of application to empirical reality or to concrete arithmetical or geometrical "manifolds," is "theoretical." As such, pure mathematics is a method of universal analysis whose objects, as well as the rules for those objects' manipulation, are sense-perceptible signs imbued with "quantitative," which is to say, "in some sense numerical," significance by mathematical theory. When Klein therefore says the symbolic nature of the modern concept of number that underlies the symbolic calculi of modern mathematics "*is identical with Viète's concept of species*," he is speaking about these things from the perspective of the philosophical clarification of their conceptual structure, and not in terms of that conceptual structure's immanent mathematical meaning. Thus, Klein is *not* saying that Viète's species is mathematically identical with the modern concept of number, when the latter is understood as that which is, for instance, realized Dedekind's theory of rational numbers as "cuts" on the number line or Cantor's theory of transfinite numbers. Rather, Klein is saying that what makes possible Dedekind's, Cantor's, or any other modern *theory* of number is precisely its point of departure from the conceptually general level of symbolically realized "general number," and with that, the "general magnitude" established for the first time in the zetetic analysis of Viète's *logistice speciosa*.

10.9 Klein's philosophical account of the symbolic transformation of ordinary numbers in Viète's interpretation of traditional numbers

There is, however, a modern concept of number that Klein *is* saying Viète's numerical innovation is identical with, namely, that of "ordinary numbers [*gewöhnliche Zahlen*]" (186/178), i.e. "numbers" as they are conceived of in "everyday [*durchschnittlichen*] understanding." According to Klein, this conception of number concerns "the presentation [*Darstellung*] of the numbers themselves [*Anzahlen selbst*]" (185/177) outside the "sphere [*Bereich*]" of the formal language of algebra, wherein numbers, in the sense of determinate amounts of determinate items, are understood "to coincide with the 'number-*sign*' [*Zahl*-Zeichen] itself" (186/178). On Klein's view, *this* conception of number, which is formed above all in non-algebraic calculation operations, is nevertheless still governed by Viète's innovative symbolic notation *and* his Analytic Art's interpretation of the "numbers" [*Anzahlen*] that are understood in terms of determinate amounts of determinate monads, "from the point of view of their symbolic presentation" (184/176).

The issue here for Klein is again the answer to the question that he poses about how it's possible for Viète's letter notation to have "numerical" significance, since, clearly, neither the letters it employs, nor the concept of "general number" these letters "embody," are numerical in the sense of a number that means a determinate amount of determinate monads. It should be stressed here that Klein's question is not about the immanent mathematical meaning of Viète's letters. Klein therefore is not questioning, i.e. calling into doubt, whether there is a structural connection between Viète's symbolic presentation of the species and some kind of numerical meaning. Rather, Klein's questioning is *philosophical*. What he is asking is how, despite the absence of a *direct* reference to numbers that the answer to the question of "how many" demands, Viète's symbolic expression of species nevertheless somehow retains a tie to this sense of the meaning of number.

Klein's answer to this question is twofold. On the one hand, Viète's rules for the *logistice speciosa* "are directly derived from 'calculations' with numbers [*Anzahlen*] of monads" (184/176), despite Viète's "contrasting them with the operational rules of the 'logistice numerosa'." Even Viète's "law of homogeneity" maintains a tie with numbers of monads according to Klein, because "Viète's law is concerned… with the fundamental fact that every 'calculation' is ultimately based on 'counting off' [*abzählen*] the basic units, which presupposes a field of *homogeneous monads*" (181/173–174). On the other hand, these "numbers [*Anzahlen*] of monads" "*are themselves interpreted as 'numbers'* [Zahlen], that is, they are understood [by Viète] in terms of their symbolic presentation" (184/176).

10.10 Klein's philosophical account of the structural ambiguity of number implicit in Viète's zetetic analysis

On Klein's view, then, Viète's "law of homogeneity" is not directly related to Adrastus' explication of Euclid V, Def. 3, despite Viète's appeal to Adrastus. This is the case for Klein because the notions of "homogeneity" are different in Euclid and Viète. Adrastus' comment referred to by Viète, that "it is impossible to know how heterogeneous magnitudes may be conjoined" (325), concerns a "sort of relation" (181/173) of size, that is, a "ratio," between homogeneous magnitudes. Homogeneity for Viète, in contrast, concerns the relation by addition or subtraction of magnitudes "as belonging to the same or the corresponding 'rung'." While in Euclidean mathematics comparisons of the ratios of heterogeneous magnitudes (e.g. lengths and planes) are made and the ratios brought into proportion, for Viète "only magnitudes of 'like genus' can be compared (i.e. can appear in the same equation) with one another." However, according to Klein, there may be a sense in which "Indirectly the 'lex homogeneorum' of Viète is identical with the statement of Adrastus, insofar as every *proportion* whose ratios are ratios in the sense of the Euclidean Definition… can be converted into an equation"

(181/176). This sense, however, presupposes that Viète "regards the theory of ratios and proportions from the outset in light of the 'theory of equations' understood as a 'theory of calculation'." Because, as we have seen, Viète's rules for relating species in the calculation constitutive of the *logistice speciosa* are derived from numbers in the sense of numbers of monads—although numbers in *this* sense are precisely what are *not* presented by the numerical sense of the species as "general number"—the "law of homogeneity" needs to stress that "the homogeneous elements of equations are *units*." This is the case for Klein, because unlike in Diophantine *Arithmetic*, where the analytic meaning of the species is inseparable from the synthesis in terms of numbers composed of monads, in the zetetic analysis of Viète's Analytic Art, "this fundamental presupposition needs to be stressed; hence the emphasis with which Viète, in contrast to the 'ancient analysts' (*veteres Analystae*), expounds the 'lex homogeneorum' as the foundation of the 'analytical art'" (181–182/174).

On Klein's view, Viète's self-interpretation of his Analytical Art, which maintains that "Rhetic and exegetic… must be considered to be most powerfully pertinent to the establishment of the art" (172/168), shows "that 'general analytic' is understood [by Viète] as nothing more than the indispensible auxiliary *means* to the solution of geometric and numerical problems." Notwithstanding this, however, Klein maintains that the philosophical structure of the modern concept of number is realized in Viète's zetetic analysis, "by means of the introduction of a general mathematical symbolism" (152/149) that "accomplishes the fundamental transformation of the conceptual foundations" behind the "operating with numbers and number signs" (152/148) manifest in the technical advances in the algebraic expositions of Stifel, Cardano, Tartaglia, etc. This is why Klein can say that even before Viète's zetetic symbolically transformed these conceptual foundations, that the "new" number concept "already guided [*leitet*]" (186/178) these technical advances. He can say this because the identification of the presentation of the numbers themselves, as determinate multitudes of units, with the number-sign, was for calculation already operationally in effect, albeit without the thorough going symbolic transformation of calculation accomplished by Viète's zetetic analysis. Moreover, this is also why Klein can say that Viète's "reinterpretation" (184/176) of ancient numbers (*Anzahlen*)—determinate amounts of multitudes of units—from the level of their symbolic presentation, which "has to this day remained the foundation of our understanding of ancient 'arithmetic' and 'logistic'," "was supported by the 'Arabic' positional system of ciphers… whose 'sign' character is much more pronounced than that of the Greek or Roman notion" (184/277). Indeed, this is also why Klein also says

> But it would be a mistake to attempt to understand the origin of the language of symbolic formalism as the final consequence of the introduction of the Arabic sign language. The acceptance of this sign language in the West *itself presupposes a gradual change in the understanding of number* [Anzahl], whose ultimate roots lie too deep for discussion in this study.

And finally, this is why, regarding the *understanding* of number, Klein can say that in Viète number is *understood* as both the general concept of number *and* at the same time as referring to entities. For instance, "two" is understood *both* as "the concept of twoness in general"[6] and "as referring to two entities." The numerical ambiguity expressed by "at the same time" here, according to Klein, is something "Modern *set-theory* first… tries to clarify." But on Klein's view, the basis of the ambiguity here extends much deeper than the paradoxes of set theory, to the late sixteenth century in fact, when Viète symbolically misinterpreted Diophantus' signs for the species—and created in the process modern algebra.

6 Jacob Klein, "The World of Physics and the 'Natural' World," ed. and trans. David Lachterman, in *The Lectures and Essays of Jacob Klein*, ed. Robert B. Williamson and Elliott Zuckerman (Annapolis, MD: St. John's Press, 1985), 1–34, here 25.

11 Stevin's *mathesis* and number

Jean-Marie Coquard

Abstract: Jacob Klein wrote about Renaissance symbolic algebra dealing with four mathematicians. Simon Stevin is the second of them, after François Viète and before René Descartes and John Wallis. In a ten-page chapter, Klein explained Stevin's theory about history, emphasizing the decimal positional system and the concepts of first and second intentions to show the importance of material cause in Stevin's concept of number. In this chapter, I build on Klein's results and show how history, mathematics and also dialectic and ethics must be thought together, in a broader perspective, in Stevin's work. This art of thinking, with a mathesis as a kernal, explains why Renaissance thinkers have to define a new concept of number.
Keyword: Simon Stevin; Jacob Klein; Number; Concept of Number; Symbolic Algebra; Mathesis; Dialectic; Arithmetic

11.1 Introduction

In this chapter, we describe Klein's arguments concerning number in Stevin's arithmetic (1548–1620) and propose an interpretation. To do so, we emphasize various aspects of Stevin's thought especially that concerning the importance of dialectic. We see, indeed, in Stevin a pre-Cartesian "*mathesis*" built on the articulation of dialectic and mathematics. We note that this "*mathesis*" extends from a theoretical and practical discussion of the Dutch language to everyday practical mathematics. In this way, we can go further than Klein and clarify why it is useful, as he began to do in his book, to understand the parts of the "Wijsen Tijd" ("The Age of the Sages") and Stevin's theory about history. That is, we then have insight into the Flemish scholar's discussions on linguistics and the understanding of his conception of unit and number.

Klein's book is made up of two parts, the first about antiquity and the second about early modern times. The early modern period begins with Viète and Stevin and we find that Klein considers Wallis to be the first to use the modern concept of number, which was in crisis when he wrote in the beginning of the twentieth century. Also we note that Diophantus, a mathematician of late antiquity, is considered pivotal[1] to this discussion. If we were to consider the early modern concept of number as the result of Diophantus' *Arithmetic* or at the most the Renaissance use of his *Arithmetic*, the

1 Jacob Klein, *Greek Mathematical Thought and the Origin of Algebra, Dover*, 1992 (1968), translated by Eva Brann (this edition will be referred as GMTOA).

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Middle Ages is not treated here. (Stevin did the same with his "Age of the Sages" whereby the Middle Ages is a "barbarous age".)

To refer again to antiquity and late antiquity as explained by Klein, we can say that one of the main problems discussed is the definition and status of number in arithmetic and logistic. Here, he deals with Plato, Aristotle and Neoplatonicians:

> We saw that the crucial difficulty of theoretical logistic as the theory of those mutual relations of numbers that provide the basis of all calculation lay in the concept of the monad, insofar as it is understood as an independent and, as such, simply indivisible object. *Aristotle's criticism obviates this difficulty* by showing that this "indivisibility" does not accrue to the monad as a self-subsisting hen, but by virtue of the *measuring character* of any such unit, be it of an aisthetic or a noetic nature. […] Nothing now stands in the way of changing the unit of measurement in the course of the calculation and of transforming all the fractional parts of the original unit into "whole" numbers consisting of the new units of measurement. Thus even fractions can now be treated "scientifically." If we disregard for a moment the fact that Plato's demand for a theoretical logistic is in fact realized within a different context, namely in the general theory of proportions, there can be no doubt that only Aristotle's conception of *mathematika* makes possible that "theoretical logistic" which suffered from the dilemma of being at the same time postulated by the *chorismos* thesis,[2] and yet precluded from realization by that very thesis. We even possess a significant document which is able to give us a concrete notion of the *type* of theoretical logistic which can be built on Peripatetic foundations – the "arithmetical" textbook of *Diophantus*.[3]

From a theoretical or philosophical rather than historical point of view, the possibility of the existence of algebra as a "theoretical logistic" is related to the definition of number. On the one hand, following this research, practical mathematics is important: number and its definition are therefore intended to find the relation with what is numbered and calculated, that is, length and so on. It demands that we understand how practices are progressively mathematized. On the other hand, there is the theoretical question of how disciplines are articulated with each other. We see here the question of the method, that is, the architecture and extension of a "*mathesis*": what rules are general enough to accommodate a larger and larger range of practices. These questions refer to Aristotle's *Posterior Analytics*, as they are read in the sixteenth century.

Jacob Klein's section discussing Stevin's number is short and quite descriptive. The sources of his text are twofold. First, he used *L'Arithmetique* of 1585 and Girard's edition of 1634 *Les œuvres mathématiques de Simon Stevin*.[4] There, Klein found

2 According to the chorismos thesis, there's a separation of all noetic formations from all that is somatic.

3 Klein, GMTOA, pp. 112–113. We can see in the question raised by Klein a neo-Kantian analogy. The double dichotomy analytic/synthetic and a priori/a posteriori looks like arithmetic/logistic and theoretical/practical.

4 Simon Stevin, *L'Arithmetique et La Pratique d'Arithmetique*, Leyden, Christophe Plantin, 1585. Albert Girard, *Les Oeuvres mathématiques de Simon Stevin de Bruges. Ou sont inserées les Memoires Mathematiques, esquelles s'est exercé le Tres-haut & Tres-illustre Prince Maurice…, augmentées par Albert Girard*, Leyden, Bonaventure & Abraham Elsevier, 1634.

some translations made by Girard himself, first published in Dutch in the *Wiscontighe Ghedachtenissen*,[5] that is, the "mathematical memoirs" of 1605 and 1608 (particularly the famous passage of the Cosmography about the "Age of the Sages"). Girard had the merit to propose a faithful translation, and his comments and contributions are well separated from Stevin's original text. Secondly, Klein followed the biography of Stevin made by Henri Bosmans.[6] The 1968 edition of Klein's book mentioned the three first volumes of *The Principal Works of Simon Stevin* published between 1955 and 1961[7] and also some panoramas of the history of mathematics, like Tropfke's.[8]

It's interesting to remark that Stevin's dialectic[9] was, and still is, considered to be a second-rate text by historians of science; and it is not even published in the five volumes of *The Principal Works*. Though Stevin explicitly mentions algebra in the 1585 treatise's preface, this dialectic is mentioned again the same year in *La Pratique d'Arithmetique*. We could say the same about the text of ethics, *Het burgherlick leven*,[10] published in 1590 and translated in the fifth volume of *The Principal Works* by A. Romein-Verschoor edited in 1966. Stevin's *Dialectike* however is interesting as we understand the use of Stevin's method, particularly the research of good axioms and we are therefore enlightened about how Stevin proceeds in *L'Arithmetique*. Did Klein know this when GMTOA was published? Even then, could he consider these texts useful to its purpose? Probably not.

In *L'Arithmetique*, Stevin wrote about numbers progressing from basic integers to a more general conceptualization for algebraic numbers. Klein demonstrates the origin of algebra on the basis of the evolutionary concept of (arithmetical) number; however, Stevin seems transparent to Klein as if there was nothing surprising about a particular sixteenth-century mathematics. It appears that Klein knew but didn't really use the knowledge at hand before Stevin in the Renaissance; I am referring particularly to a reading of Euclid concerning relations between arithmetical and geometrical books, or unification in the education of liberal arts, trivium and quadrivium. We could say that Klein is too arithmetical, and this effects his reading of Stevin whose algebra should be understood in a broader perspective.

Following Klein's chapter, we will begin by Stevin's "Age of the Sages" (11.2). This part will show us that the question of Stevin's number must be understood in a broader perspective, so we will discuss Stevin's "*mathesis*" (11.3). Then, we will go back to Stevin's number and follow his texts (11.4.1 and 11.4.2) and Klein's book (11.4.3). We will then give some of our interpretations (11.4.4).

5 Simon Stevin, *Wiscontighe Ghedachtenissen, inhoudende t'ghene daer hem in gheoeffent heeft den Doorluchtischsten Hoochgheboren Vorst ende Heere, Maurits, Prince van Orangien…*, Leyden, Jan Bouwensz., 1605 and 1608.

6 Henri Bosmans, "Stevin", *Biographie Nationale*, vol. XXIII, 1924, pp. 884–938.

7 Ersnt Crone, E.J. Dijksterhuis, R.J. Forbes, M.G.J. Minnaert, A. Pannekoek, *The Principal Works of Simon Stevin*, Amsterdam, C.V. Swets & Zeitlinger. The volumes about mathematics are II A and II B, edited by D.J. Struik in 1958. They will be referred as PW.

8 Johannes Tropfke, *Geschichte der Elementarmathematik III*, 1903.

9 Simon Stevin, *Dialectike ofte bewijsconst*, Leyden, Christophe Plantin, 1585.

10 Simon Stevin, *Vita Politica. Het Burgherlick Leven*, Leyden, Franchoys van Ravelenghien, 1590. This text was recently published and commented by Pim den Boer in Dutch and by Pim den Boer and Catherine Secrétan in French; see *De la vie civile*, ENS Editions, 2005.

11.2 The "Age of the Sages" theory

In 1608, Stevin published the second part of his *Wiscontighe Gedachtenissen*, the mathematical memoirs made for Maurice of Nassau. Before dealing with geography, he wrote some pages about history and the renewal of a golden age where knowledge and experience would be complete. Stevin defines the "Age of the Sages" in the following way:

> 6th definition.
>
> AGE OF THE SAGES we call that time in which exceptional learning was to be found among men, a fact which we perceive with certainty from certain signs, but without knowing among whom or when.[11]

By writing "*wijsentijd*", Stevin points towards an age where people are sage, and not towards a wise era. Girard translated here "Siècle sage", and Klein followed him "Wise Age", which is a misinterpretation. In this way, Girard and he miss the ethical dimension of this renewal. These six signs [teycken] Stevin considers correspond to the use of some disciplines, which must be reactivated:

- astronomy – the *auctoritas* on that matter are Ptolemy and Hipparchus; they have written what must be regarded as remnants of what had existed in the *wijsentijd*, hermetism as mentioned here;
- arithmetic and algebra – following the numerals with the tenth progression for the first Arabic books and Diophantus for the second. Diophantus is recent, "jonck",[12] according to Stevin, so algebra is a complete novelty, while Euclid is only the recipient of the *Age of the Sages*;
- geometry – Euclid received and kept alive the "systematic order" in the method of description of mathematics;
- height of clouds – an Alhazen and Nonius matter, it's an enquiry into the form of the world and of nature's hidden properties;
- alchemy – in reference to Hermes Trismegistos; this is an "inexhaustible source of wisdom" (this qualification reminds the source for Algebra in *L'Arithmetique*, "admirable Reigle d'Algebre, l'inexhauste fontaine d'infinis théorèmes arithmetiques").[13]
- magic – to paraphrase Stevin; magic is diligently practised by certain nations with mathematical knowledge as to the causes of what seems frightening. We remark that, like Girard in 1634, Klein doesn't mention this sixth sign.

We see that there is an ethical and political project behind this *Age of the Sages* theory: to understand these disciplines allows us to be wise. Mathematics allows us to

11 PW III, pp. 600–601. *Wiscontighe Gedachtenissen*, Tweede Deel des Weereltschrifts, Vant Eertclootschrift, Eerste Bouck des Eertclootschrifts, van syn bepalinghen int ghemeen, p. 9. Klein follows Girard French translation, Albert Girard, *Les œuvres mathématiques de Simon Stevin*, Géographie, p. 106.

12 Here, Girard translated in French "Diophantus Is Jonck" by "Diophante est moderne", p. 108; PW III in English by "Diophantus Is No Ancient Writer", p. 599.

13 *L'Arithmetique*, Problème LXXXI, p. 398.

understand the world and helps us to act usefully.[14] What seems strange shouldn't be frightening, because the world is perfectly ruled, and the only miracle is that there is no miracle, "wonder en is gheen wonder";[15] there is only ignorance.

Stevin explains that the renewal he wants demands what Klein calls a "general plan" "in four articles":

- many observations – all over the world and beyond the nations;
- the use of mother tongue;
- each language has its own virtue, even if Flemish is – of course – better than other languages;[16]
- a scientific presentation using the right order ("le bon ordre en la description et instruction des arts").[17]

This last article is itself divided into five chapters. He discussed there Euclid's style (with its different members of the resolution of a problem/proposition), the role played by definitions, the dichotomy, the anaphora ("the same concepts should invariably be referred to by the same terms"). This explains why Stevin says algebra is the "rule of false algebraic numbers", for example) and, in the end, that is the articulation between theory and practice.

We see that the "Age of the Sages" theory enlightens his project and specifically the role played by mathematics. Stevin himself explains to us why this theory is interesting, his conception of the number when it comes to the difference between 0 and 1, and why we must not follow Greek mathematicians:

> The reason why we here discuss this point so seriously is that the definition of unity as the point of number is evidence, among other things, of the Age of the Sages which existed when the point was drawn and looked upon as such; and also of the Age of the Ignorant which has existed since then producing as imperfect arithmeticians as the definition of a part of a magnitude as a point of the magnitude would testify to imperfect geometers. It may also be noted that in this Age of the Sages many arithmetical operations were performed with extraordinary ease by means of computation based on the tenth progression. In order to explain this more fully, it is to be noted that while some years ago I described the Tenth and imagined that it might be used with great ease in the division of sines and arcs according to the tenth progression; and while thereafter I described this method properly in such a way as in its place a chapter is to be made about it in the subsequent book on astronomy, with the succinctness that will appear there, I perceived

14 Pierre Hadot, *La philosophie comme manière de vivre. Entretiens avec Jeannie Carlier et Arnold I. Davidson*, Le livre de poche, 2001; Giovanna Cifoletti, "L'utile de l'entendement et l'utile de l'action. Discussion sur l'utilité des mathématiques au XVIe siècle", Revue de synthèse: 4e sér., nos 2-3-4, avr.-déc. 2001, pp. 503–520.

15 This is Simon Stevin's motto.

16 Stevin explained why, in 1586, for grammatical reasons and following a particular conception of the relation between language and reality, North Hollandic is better than any other language. Marijke van der Wal, "Simon Stevin, observateur et locuteur de la langue", in *Simon Stevin (1548-1620): l'émergence d'une nouvelle science*, Bibliothèque royale de Belgique, Brepols, 2004, p. 171.

17 Klein, GMTOA, pp. 188–190.

> afterwards that this had already been done before me, or at least seemed to have been done in ancient times, which I think was the Age of the Sages.[18]

He justifies then why *L'Arithmetique*, published in 1585, is in many ways a small revolution, as a return to the origin. According to us, Klein is right when he begins his article on Stevin with the "Age of the Sages". We see indeed then why algebra is always embedded and must be understood in a broader "art of thinking",[19] (that is) a consistent articulation between different faces of the same theoretical and practical epistemology, including linguistic, logical, ethical, physical and of course mathematical points of view.

In *Greek Mathematical Thought and the Origin of Algebra*, the chapter about Diophantus is followed by the one on Viète's reinterpretation of Diophantus. There is a shortcut excluding the Middle Age, which is not trivial because Klein used the medieval concepts of *intentio prima* and *intention secunda* for his argumentation.[20] Klein begins his chapter, "The concept of 'number' in Stevin", by contrasting him with Viète in length as developed in the chapter before. It's interesting that he chose to write in this non-chronological order: Viète is more precise in his philosophy and once we have understood his concept of number, we can return a decade earlier to explain Stevin's theses.

One of these elements of opposition, according to Klein, is the fact that Viète is "on principle conservative", whereas Stevin "consciously breaks with the traditional form of science and puts his "practical" commercial, financial and engineering experience to the service of his "theoretical" preoccupation, just as, conversely, his "theory" is put to use in his "practical activity""[21] (this is explained in the last "chapter" of the "Age of the Sages"). Klein nuances direct the opposition, explaining that Stevin is not so detached from the past since he "is possessed by the idea of a 'renewal'. He casts it into the specialness of a 'wise age', a golden age in Antiquity that must be brought back".[22] Klein's third chapter on Stevin records this history.

11.3 Stevin's *Mathesis*

Despite his introduction with the "Age of the Sage" theory, Klein seems to underestimate its role in Stevin's thought in order to understand his conception of number. We try now to give some elements of this *mathesis*.

18 PW III, pp. 600–601.

19 Giovanna Cifoletti, "The Art of Thinking Mathematically", *Early Science and Medicine*, vol. XI, n. 4, 2006.

20 GMTOA, p. 174. See also Alain de Libéra, *La querelle des universaux*, Seuil, 1996, p. 283, sq. The medieval scholastics developed *prima* and *secunda intentiones* following a lecture of Aristotle's *Posterior Analytics*, Porphyrus' *Isagoge*, Avicenna and Averroes… See in p. 180 of de Libéra's book a discussion on the difference between Avicenna's, Husserl's and Brentano's conception of intention. For the Klein's use of "intention", we can refer to Burt C. Hopkins, *The Origin of the Logic of Symbolic Mathematics. Edmund Husserl and Jacob Klein*, Indiana University Press, 2011, particularly §105, p. 310.

21 Klein, GMTOA, p. 186.

22 Klein, GMTOA, pp. 186–187.

11.3.1 Arithmetic and geometry

In 1540 in *Methodus facilis*, Gemma Frisius gives to the rule of three the name "regula proportionum", rule of proportions. It wasn't common in the beginning of the sixteenth century to use proportion in an arithmetical context; "proportion" is a priori a word of geometry. A lot of mathematicians following the same path will multiply the parallelism between geometry and arithmetic using Campanus-Zamberti's version of Euclid by finding some similarities in arithmetical and geometrical books. Tartaglia and Clavius were some of them.[23]

The articulation between arithmetical and geometrical numbers stated in *L'Arithmetique* (see our paragraph IV.2) is also an articulation between arithmetic and geometry as disciplines and as Stevin describes it in 1583 in *Problematum Geometricorum Libri V*:

> Second book of the regula falsi of continuous quantity. What the Regula falsi is Since we have arranged geometry (which we shortly hope to publish) in a Method similar [methodo similem] to the method of Arithmetic [Methodum arithmeticae] (which the natural order [naturalis ordo] of things seems to require because of the great agreement between continuous and discontinuous quantities: we shall deal with any kind of magnitude such as a line, a plane figure, a solid and by the four operations, viz. Addition, Substraction, Multiplication, and Division and also by the rules viz. of proportions, etc.); a certain Problem presented itself also in due time where by a false position we could find by Geometrical means the true solution sought. Therefore, in order to render the correspondence between continuous and discontinuous quantities, the more evident (for in Arithmetic there is a certain common rule which is called regula falsi), we have called it the Regula falsi of a continuous quantity not because it teaches false things but because knowledge of true things is arrived at by a false position.[24]

The vocabulary of dialectic (methodo, ordo, …) is present in this passage and it is indeed this discipline which has the pretention of giving a common method to every art. We can read the geometry/arithmetic conciliation in Stevin's texts in the definition of quantity proposed in the text of dialectic:

> Definition. Quantity is what can be partaken in parts. Explanation. So the magnitude (which has three species: solid, surface, line) and the number, because they can be partaken in parts, are called quantities.[25]

23 Sabine Rommevaux, *Clavius, une clé pour Euclide au XVIe siècle*, Paris, Vrin, 2006; Antoni Malet, "Renaissance Notions of Number and Magnitude", *Historia Mathematica*, vol. 33, n. 1, 2006, pp. 63–81; Odile Kouteynikoff, François Loget, Marc Moyon, "Quelques lectures renaissantes des Eléments d'Euclide", in Evelyne Barbin and Marc Moyon, *Les Ouvrages de mathématiques dans l'histoire. Entre recherche, enseignement et culture*, PULIM, 2013, pp. 13–28.

24 Simon Stevin, *Problematum Geometricorum Libri V*, Anvers, Jean Bellere, 1583, book II. English translation in PW II A, p. 207.

25 "Definitie VII. Menichvuldicheyt, is die deelelick is in deelen. Verclaringhe. Als Grootheyt (diens drie specien Lichaem, Plat, Linie) ende Getal, want sy deelelic sijn in deelen, worden Menichvuldicheyt ghenoemt", Simon Stevin, *Dialectike ofte bewijsconst*, Definition 7, p. 10.

Not only the rapprochement between geometry and arithmetic is clear, but the existence of the word used here for quantity, "menichvuldicheyt" (which refers to the character of abundance or multitude, multiplicity or to the product of multiplication),[26] with a unique definition shows that there is a concept of general quantity generalizing the number in arithmetic and the geometrical magnitude. "Menichvuldicheyt" is strictly neither "ghetal", nor "grootheyt".

11.3.2 Mathematics and dialectic

We chose our previous examples to emphasize the role of dialectic in Stevin's "art of thinking" and his mathematical texts. In Gemma Frisius' handbook, we quoted and also found this parallel: the operations with numbers (addition, subtraction, multiplication and division) are compared with the four dialectical species (example, enthymemes, induction and syllogism).[27]

Stevin was enrolled in the University of Leiden, 1583, wrote his dialectic and published *Tafelen van Interest*, in 1585, a text which for the first time gave tables together with their method of use; this allowed merchants to decide for themselves about lending money. As usual, Stevin began with definitions and continued with propositions and problems. We find the same determination to use the terms of dialectic in a mathematical text (here underlined):

Proposition I. Given the Principal, the time, and rate of simple and profitable interest: find the interest.

Note. It is to be noted that just as discontinuous proportion consists of 4 <u>terms</u> [termijnen], of which, when three are known, the fourth becomes known therefrom. [...]

Example 2. 27 lb gives 14 lb of simple interest in 4 years; what does 320 lb gives in 5 years?

Procedure. Since these five given <u>terms</u> have not been <u>arranged</u> [ghedisponeert] in the previous way, they have to be <u>arranged</u> [disponeeren] in the following way: the product of the two first <u>terms</u> gives the <u>middle term</u> ['tmiddel termijn]; what does the product of the two last <u>terms</u> give? That is: 108 lb (for that is the product of the two first <u>terms</u>, to wit 27 and 4) gives 14 lb (that is the <u>middle term</u>); what does 1600 give (for that is the product of the two last <u>terms</u>, to wit 320 and 5)? This is 207 11/27 lb.[28]

The translation does not show all of the relations between dialectic and arithmetic. The verb *disponeeren* (and its past participle *ghedisponeert*) makes reference to the *dispositio*, the second part of the Ciceronian *oratio* where the arguments found in the first part, the *inventio*, are collated and arranged. The expression middle term, *middel termijn* in Dutch, is of course a parallel to the middle term of a syllogism.

The translation of this handbook in French, published by Christophe Plantin in 1585 in *La Pratique d'Arithmetique*, is more careful of the spatial organization on the page of the dialectical "dispositio". The presentation made for finding interest is the same for Gemma Frisius' and for finding a fourth proportional: on a line, the three first terms are put in order; the empty place is marked with some points. The

26 Marjolein Kool, "De rekenkundige termen van Simon Stevin", *Scientiarum Historia*, vol. 18, 1992.

27 Gemma Frisius, *Arithmeticae practicae methodus facilis*, Anvers, Gregorio Bontius, 1540, pars prima.

28 *Tafelen van Interest*, PW II A, p. 35.

use of the rule of three is then more explicit.[29] In *Dialectike*, the same is made of the invention of the middle term in syllogisms.

This *disposition* about fourth proportional is useful in a lot of matters. What is true of theory is also true in practice when numbers are related to substantial things as here with a converse rule of three:

Example 1. 2 ells of canvas cost 3 lb, how many ells will be given for 6 lb. Construction. We will put the terms given in order, but instead of the unknown term, we will put 0: 2, 3, 0, 6. To find the third term, unknown, we will say with the opposite of the 82 definition, 3 gives 2, how many 6? which gives as an answer 4 ells.[30]

This investment in proportion furnishes Stevin with a good parallel between the mathematical language and the Flemish one, by means of dialectic. He called "dialectical proportion" the equivalent of proportion in language:

DEFINITION XXVI.

A dialectical proportion is a comparison of two similar reasons.

EXPLANATION.

The mariner is similar to his boat as the king is to his kingdom.

This is a dialectical proportion, in which the first reason compared is the mariner to his boat, the second the king to his kingdom, we understand the proportional terms as, mariner, boat, king, kingdom. By analogy with the reason of arithmetical proportions, which are said double, triple, etc., it could be said to the right government, viz the same manner a mariner must well govern his boat, similarly a king must well govern his kingdom.[31]

And then in L'Arithmetique:

Proportion, to discuss is somewhat general before we come to the particular is the similitude of two equal ratios. Ratio is the comparison of two terms of a similar kind of quantity. And if all the terms of a proportion were magnitudes, it would be a geometrical proportion. But if they were all numbers, the proportion would be an arithmetical one, and if they were all harmonic sounds, it would be a harmonic proportion. Similarly, if the terms are parts of predication or proposition, the proportion is a dialectical one. Thus every proportion receives its name in conformity with the nature of its terms.[32]

Stevin gives to his dialectic the key role of distinction between truth and false, the verification of the good interaction of theory and practice and an art of writing in

29 Simon Stevin, *La Pratique d'Arithmetique*, Plantin, Leyde, 1585, p. 63.

30 "Exemple 1. 2 aulnes de toille coustent 3 lb, combien des aulnes se donneront pour 6 lb? Construction. On mettera les termes donnez en ordre, mais au lieu du terme incognu, on mettera 0, en ceste sorte: 2 . 3 . 0 . 6 Or pour trouver le troisiesme terme incognu, on dira par la renverse proportion de la 82 definition, 3 donne 2, combien 6? faict pour solution (par le precedent premier exemple) 4 aulnes; Ou autrement par alterne proportion de la 83 definition, 3 donnent 6, combien 2? faict comme dessus 4 aulnes". Simon Stevin, *La Pratique d'Arithmetique*, Plantin, Leyde, 1585, pp. 36–37.

31 "Definitie XXVI. Dialectike Proportie, is de verghelijckinge van twee gelijcke Redenen. Verclaringhe. Ghelyck den Schipper tot sijn Schip, Alsoo den Coninck tot sijn Coninckrijck. Dit is een Dialectike Proportie, diens eerste vergheleken Reden, is, Schipper tot Schip; dander Coninck tot Coninckrijck, inder voughen dat der Proportien Termijnen, zijn, Schipper, Schip, Coninck, Coninckrijck; Wederom ghelijck de reden der Arithmetiker proportien, gheseyt worden Dobbel, Drievuldich, ofte dierghelijcke, so meughen dese geseyt worden Goede Regiering, dat is, gelijck den Schipper sijn Schip wel moeten regieren, also oock den Coninck sijn Coninckrijck". Simon Stevin, *Dialectike ofte bewijsconst*, Definition 26, p. 30.

32 Simon Stevin, *L'Arithmetique*, pp. 56–57, PW II B, p. 546.

Dutch for the "discovery of the mysteries hidden in nature". According to Stevin in *La Pratique d'Arithmetique*:

> The effectiveness of the order of disciplines is such that we learn easily and with pleasure what we did otherwise hardly and boringly, what we treated in general in our Dialectic. It has seemed good to me to say something of the order of the Practice of Arithmetic.[33]

The dialectical *dispositio* is closely related to the mathematical techniques. Of course, dialectic is also "the art which leads to all the other arts", by the research of a good "art of writing" and of a general method.

Reading *L'Arithmetique* with the *Dialectike*, we understand Stevin's "*mathesis*" as an articulation of mathematics and dialectic, particularly in the explanation of a method built on linguistic investigations related to treatises written in a "mathematical style" according to Euclid. We can notice that Jan van Hout's preface of the first Dutch translation of Euclid, made by Jan Pieterszoon Dou, paraphrases Stevin's *Dialectike* preface.[34] The last article of the "Age of the Sages" about the order, in the text of 1608, didn't refer explicitly to dialectic; perhaps this is the reason why Klein didn't speak of it, but it is indeed what is at stake here.

11.3.3 Grammar and mathematics

As Klein pointed out, the use of the mother tongue is important and Stevin tries to show the reasons why Dutch is a good language for the sciences. The analogy made in Stevin's *Geography* of *words of one or more syllables* and simple and complex things of the world built within the elements is another example of this procedure of analogy; here, it is between linguistics (phonology and grammar) and physics (as a theory of Nature). In Dutch, says Stevin, monosyllabic words (as short sounds) represent basic elements. By concatenation, as it is possible in Dutch, monosyllabic words put together can produce bigger words representing more complex things. Stevin then explains what must be the rules of such compositions, and why Dutch is the only language which respects these rules and concludes with the superiority of his language. Stevin built an articulation between logic and grammar and related this articulation with mathematics:

> It is further to be noted that they have also done this in the elements of grammar, i.e. in the letters, all of which they denote by monosyllables, which is certainly nearer to the highest perfection than the contrary; for just as in geometry it would be absurd to consider the point, the element of magnitude, greater than magnitude itself, in the same way it is also improper in grammar for the element to consist of more syllables than that which is made of several elements.[35]

33 "Vu que l'efficace de l'ordre des disciplines est telle, que l'on apprend par icelle facilement et à plaisir, ce qui autrement ne se fait qu'a grand labeur, et ennui, de quoi nous avons traité en général en notre Dialectique, il m'a semblé bon de dire ici en particulier de l'ordre de la Pratique d'Arithmétique", Simon Stevin, *La Pratique d'Arithmetique*, Plantin, Leyde, 1585, Adresse au lecteur, p. 3.

34 Jan Pieterszoon Dou, *De ses eerste boecken Euclidis, van de beginselen ende fondamenten der geometrie*, Willem Janszoon Blaeuw, Amsterdam, 1626.

35 PW, IV, pp. 80–93.

There is obviously in Stevin's thought an articulation between trivium and quadrivium.

11.3.4 Method, practical mathematics and **Mathesis**

This explains why Renaissance mathematicians explored the notion of method and, for some of them, published dialectics. The vernacular language has a great importance because it contains a logic different from the scholastic one which is supported by Latin. Giovanna Cifoletti has shown in the French tradition of algebra (Jacques Peletier du Mans, Pierre de La Ramée, Guillaume Gosselin, etc.) how "Peletier's algebraic program is connected to his theory of rhetoric", and how the promotion of French as a scientific language, embedded within the new dialectic of Lorenzo Valla, Rudolph Agricola or Erasmus, modified the way these mathematicians structured their texts. At the end of this sixteenth-century tradition, Gosselin makes the analogy of the rhetorical *quaestio* with the algebraic *aequatio*.[36] Where Vieta recomposed the *katholou pragmateia* in a *mathesis universalis* (Klein dealt with this question in the last part of his chapter on Vieta),[37] Stevin is more scholastic and dependent on a linguistical logic. In *Dialectike*, he explains, indeed, his own "method" in four points:

> In the first place, it is necessary to make a good distribution to know the parts or species of given sets, or genera for we can't justify singularities of a science without them. Secondly, we partake according to the dichotomy, always dividing into two parts; but we can divide into the lesser parts as possible, that is into three, into four... when it can't be done adequately following opportunities imposed on us. Thirdly, the parts, the species, reciprocal or contraries or such other must be put on the same level when we can find them. Who would differentiate animals, saying this one is a horse and this one a mare makes a mistake, since every mare is a horse. Fourthly, parts or species, being sets, must be given into account by their effects (whereby for human beings the division is in two parts as with man and woman); or powers, as the species of man can be Jan, Pieter, Jacob and so on with every word and so on, viz; we can understand it otherwise by taking everything in its singularity, going farther until it is inseparable.[38]

36 Giovanna C. Cifoletti, *Mathematics and Rhetoric. Peletier and Gosselin and The Making of the French Algebraic Tradition*, dissertation presented to the faculty of Princeton University, 1992. See also "The Art of Thinking Mathematically", *Early Science and Medicine*, vol. XI, n. 4, 2006 and particularly the article "From Valla to Viète: The Rhetorical Reform of Logic and Its Use in Early Modern Algebra", pp. 369–423. This kind of study reassesses humanism and science, even when the logic of humanism and ciceronian style are probabilistic. Ann Blair and Anthony Grafton, "Reassessing Humanism and Science", *Journal of the History of Ideas*, vol. 53, n. 4, 1992, pp. 535–540.

37 Klein, GMTOA, 11, C, 3, pp. 178–185.

38 "Ten eersten ist noodish om wel te verspreyden, datmen hebbe kennisse der Deelen, ofte Specien, des ghegheven Heels, ofte Geslachts, sonder welcke wetenschap, men niet besonders in desen en can uytrechten. Ten tweeden, salmen de Tweevuldige Verspreyding, als Twit, altijt natrachten, maer die haer niet bequamelic ontmoetende, so machmen naer de minstvuldichste sien, dieder met goede ghelagentheyt vallen wil, als drievuldige, viervuldige, &c. Ten derden, dat de Deelen ofte Specien, malcanderen so contrarie, ofte vanden anderen so wijt verscheyden zijn, alsmense vinden can, want dat yemandt willende Onderscheyden de Gedierten, seyde sommige te wesen Peerden, anderen Merien tis een ongeschicte Verspreyding, angesien alle Merie, Peert is. Ten vierden, dat de Deelen ofte Specien, Heelende zijn, ende dat matter Daet (als Menschen twee volcommen Specien zijn Man ende Vrouwe)

The aim is to construct a dichotomical table which gives the structure of a text and articulates the concepts of a discipline. This unique "method" is general enough to be applied to every art (*const*) or science (*wetenschap*). As an example of the method, Stevin "reconstructs" dialectically the text *Tafelen van Interest* published two years earlier to show explicitly its structure.[39]

There is, parallel to the articulation of dialectic – mathematic, a movement of generalization by mathematization, begun with Pacioli's *Summa* or in the practical quadrivium of Oronce Finé's *Protomathesis*.[40] A new (practical) encyclopaedia, based on algebra, is hence possible and developed all along the sixteenth century. This requires a methodological conciliation, as we have seen in *Problematum Geometricorum Libri V*, of the arithmetical number and of the geometrical magnitude, by their analogy with proportion.

In the progressive approach in *L'Arithmetique* beginning with arithmetical numbers and leading to algebraic numbers, algebra is presented in the same way as the research of a "fourth algebraic proportional". Stevin explains that an equation is the "invention of the fourth proportional in quantities" (an expression which reminds us of the invention of the middle term); and it's better to use an expression which indicates clearly what a thing is rather than a word which lets us think that it is a singular matter, whereas it is common in arithmetic. In the following problem, to solve the equation of the first degree, we would write 2x = 6 and we would say "2 unknown are 6, how many 1 unknown"?

Problem LXVII. Let's be given three numbers, the first (1), the second (0) and the third an algebraic number to find their fourth proportional term.

Explanation of the given: Let us be given these three terms: the first 2 (1), the second 6 and the third 1 (1). Explanation of the required. We have to find their fourth proportional term. Construction. We will divide the 6 of the second term by 2 of the first term (since the number of the third term is 1, it won't be necessary to do the multiplication of the third and second term, which is 6 (1)), it gives 3. I say that 3 is the fourth proportional term required. Demonstration. Since we say by this problem 1 (1) is 3, according to 66th problem, 2 (1) will be 6, put under each term its value:

2 (1)	6	1 (1)	3.
6 6	3	3.[41]	

ofte Machtelick, als de Specien des Mans, zijn Jan, Pieter, Jacob, ende so voorts, met welcke woorden, Ende so voorts, ofte yet dergelijcke cracht hebbende, wy alle d'ander verstaen, die ons int besonder onnoemelic sijn, want hoemen maer naect het Onscheydelicke, hoemen naerder 'toneyndelicke comt. Twelc angemerct laet ons commen tot onse voorgenomen Werckinge". Simon Stevin, *Dialectike ofte bewijsconst*, p. 56. We can think of course of Descartes' *Discours de la Méthode*, book II, published 50 years later. See also Neal W. Gilbert, *Renaissance Concepts of Methods*, Columbia University Press, 1960 and Ann Blair, "Humanist Methods in Natural Philosophy: The Commonplace Book", *Journal of the History of Ideas*, vol. 53, n. 4, 1992, pp. 541–551.

39 A description of this passage is made in Jean-Marie Coquard, "Mathématiques et dialectique dans l'oeuvre de Simon Stevin: l'intérêt des séries de problèmes", in Alain Bernard (ed.), *Les séries de problèmes, un genre au carrefour des cultures*, EDP Sciences, Labex HASTEC, 2015.

40 Luca Pacioli, *Summa de arithmetica, geometria, proportioni et proportionalita*, Venise, Paganini, 1494. Oronce Finé, *Protomathesis*, Paris, Simon de Colines, 1530. These texts were studied in Cifoletti's EHESS seminar "L'algèbre comme art de penser entre cosmographie et mathématiques du négoce".

41 "Probleme LXVII. Estant donnez trois termes, desquels le premier (1), le second (0), le troisiesme nombre algebraique quelconque: Trouver leur quatriesme terme proportionel. Explication du donné. Soient donnez trois termes selon le probleme tels: le premier 2 (1), le second 6, le troisiesme 1 (1).

When he defined what an equation is, speaking of "invention of fourth proportional in quantities", Stevin made an interesting complement:

But what they call equation doesn't consist of the equality of absolute quantities but in equality if their values. This proportion consists similarly of the value of quantities, as it is common to common substantial things. For example, let us say an ox is the value of 2 sheep with 8 lb, ergo 1 sheep values 4 lb, which are four proportional terms, not according to the quantity by which the product of the first and the fourth is not equal to the product of the middle ones, but according to the value. Because as 16 lb value of the ox, has 16 lb of value of 2 sheep with 8 lb, then 4 lb is the value of 1 sheep, 4 lb is the value of the fourth term, the four are proportional terms. We put them in order, for greater evidence:

1 ox	2 sheep + 8 lb	1 sheep	4 lb
16 lb	16 lb	4 lb	4 lb.

The same extends to the quantities, since when we say 1 (2) equals 2 (1) + 8, ergo 1 (1) is 4, these are four proportional terms according to their value, for which the product of the extreme term is only equal to the product of the middle ones. Their disposition, following the preceding one, is:

1 (2).	2 (1) + 8.	1 (1)	4.
16	16	4	4.[42]

We see with this example that the way of dealing with abstract number and number of sheep is exactly the same. We can then state that there is a unique way of calculation with the number of objects (number as number of something), with abstract number and with algebraic number.

11.3.5 Ethics and mathematics

These elements remind us of why Stevin wrote about the "Age of the Sages", and perhaps Stevin was more accurate than Klein's opinion of him. Stevin's mathematics

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Explication du requis. Il faut trouver leur quatriesme terme proportionel. Construction. On divisera le 6 du second terme, par 2 du premier terme (car puis que le nombre du troisiesme terme est 1, il ne sera besoing de faire la vulgaire multiplication du troisiesme & second terme, qui serait 6 (1)) donne quotient 3. Je dis que 3 est le quatriesme terme proportionel requis. Demonstration. Puis que nous disons par ce probleme, 1 (1) valoir 3, doncques par le 66 probleme, 2 (1) vaudront 6, mettons doncques sous chascun terme sa valeur en ceste sorte: 2 (1) 6 1 (1) 3. // 6 6 3 3". Simon Stevin, *L'Arithmetique*, Plantin, Leyde, 1585, p. 284.

42 "Mais comme cela qu'ils nomment équation ne consiste point en égalité des quantités absolues, mais en égalité de leur valeur. Ainsi consiste cette proportion en la valeur des quantités, comme le semblable est vulgaire, aux communes choses corporelles. Par exemple, un bœuf vaut 2 moutons avec 8 lb, ergo 1 mouton vaut 4 lb, lesquels sont quatre termes proportionnels, non pas selon la quantité, en respect de laquelle le produit des extrêmes n'est point égal au produit des moyens, mais selon la valeur. Car comme 16 lb valeur du bœuf, a 16 lb de valeur de 2 moutons avec 8 lb, ainsi 4 lb valeur de 1 mouton, a 4 lb valeur du quatrième terme, lesquels termes proportionnels, nous mettrons en ordre, pour plus grande évidence, en cette sorte: 1 bœuf 2 moutons + 8 lb 1 mouton 4 lb // 16 lb 16 lb 4 lb 4 lb. Le même s'entend aussi des quantités, car quand nous disons 1 (2) est égal ou vaut 2 (1) + 8, ergo 1 (1) vaut 4, ce sont quatre termes proportionnels, mais au respect de leurs valeurs, desquelles le produit des extrêmes est seulement égal au produit des moyens. Leur disposition conforme à la précédente est telle: 1 (2). 2 (1) + 8. 1 (1) 4. // 16 16 4 4". Simon Stevin, *L'Arithmetique*, Plantin, Leyde, 1585, pp. 265–266.

is included in a broader epistemology, and we see here articulations between the logic of language, arithmetic, physics and ethics. We can compare this epistemology with the ones of Hendrik Spieghel and Dirck V. Visscher. Spieghel published a dialectic in Dutch the same year as Stevin, in 1585; Visscher the second is a polemist who explained in 1586 in Dutch that science – *wetenschap* – is a special kind of knowledge – *kennisse*.

> For Coornhert, this difference between "*kennisse*" and "*wetenschap*" is essential, since only truthful knowledge or "*weten*" can make it possible for man to act truly virtuously, in accordance with God's laws. In Coornhert's paradigm, this "*weten*" is synonymous to both "*ware kennis*" [true knowledge] and "*wetenschap*" and is also closely related to the first cardinal virtue of "wisdom". Just as in one of the Zedekunst's main sources, the Nicomachean Ethics, Coornhert does not aspire to a mere theoretical (sophia), but to an applied wisdom – the phronêsis of ancient tradition.[43]

We can say, with Marijke Spies, that the rhetorical reform of dialectic has philosophical, political and religious consequences:

> What he and Visscher were propagating was no small thing: rhetoric, the discipline of 'unlearned' critical rationality based on the light of reason given to every human being, was conceived as the sole foundation of truth and morality, and, therefore, of a peaceful society.[44]

We could quote Jean Bodin's, 1577, *Six livres de la république*, where proportions are the key to building a good Republic as well as the model of harmony in a concert. Stevin gives an example of music in *L'Arithmetique*: when proportionality is not well-defined, there's no consistency between theory of music and arithmetic and it remains mysterious.[45] Conversely, our mind knows when two sounds are in harmony when we hear them and that proves the harmonic is in proportion to what it is.[46] Further, in this late sixteenth century, it is not a coincidence that Stevin created the word "wiskunde" (a Hollandic writing which should be translated into "wijsconst" in Flemish) for "mathematics", "bewijsconst" (art of demonstration) for dialectic: the originality of Stevin, with his "wijsen tijd" project, proposes a mathematization for the research of truth. Seeking the foundations becomes possible and thanks to dialectic-mathematics, it allows a good description of the world from observation. We see here a renewal of Ptolemy's project and the foundation of a mathematical ethics[47]:

43 Julie Rogiest, "Knowledge and Auctoritas in Coornhert's Zedekunst", p. 141. This *phronesis* of *Nicomachean Ethics* is read with the Stoa *phronesis*, probably the one used by Spieghel. See also Donald R. Kelley, ""Second Nature": The idea of Custom in European Law, Society, and Culture", in Anthony Grafton and Ann Blair (eds.), *The Transmission of Culture in Early Modern Europe*, University of Pennsylvania Press, pp. 131–172.

44 Marijke Spies, *Rhetoric, Rhetoricians and Poets*, pp. 67–68.

45 Simon Stevin, *L'Arithmetique*, pp. 56–57.

46 Simon Stevin, *La Pratique d'Arithmetique*, p. 17.

47 Alain Bernard, "The Significance of Ptolemy's Almagest for Its Early Readers", *Revue de synthèse*, tome 131, 6^{e} série, n°4, 2010, pp. 495–521. Ptolemy's ethics has a reception in Renaissance with Regiomontanus in fifteenth century. We can therefore inscribe Stevin in a path joining Regiomontanus and

that to find good definitions and good axioms has a direct correspondence with good practice.

It is not so surprising, then, to find in Stevin's Ethics, *Het burgherlick leven*, two political axioms ("axioma politicum", "burgherlick reghelen"), obtained by the induction with history as it is in jurisprudence.[48] We are reminded of Stoicism's applied wisdom (Zenon read in Diogenes Laertius) that to be wise is to follow one's own sentiment and conscience shaped by great knowledge which puts order in our mind.[49] So we see a good definition gained from good practices which is a dialectical matter. It is crucial that mathematics as an art of writing has an ethical dimension.

11.3.6 Induction and axioms

Following Klein, symbolic algebra appears when the number is relational (the number is a number of something) and when algebraic numbers (number as second intention) are read as every common number (number as first intention), that is, when the operations with algebraic numbers are the same operations with any number. These considerations lead us to the way Stevin considers the exercise of conceptualization, that is, his "psychology". The very first words of *L'Arithmetique* of 1585 are:

> Since arithmetic (that is common with the other arts) is explained by words as signs of the affection of the soul, which are denoted with scriptures...[50]

These words are a reference to the first ones of Aristotle's *On interpretation*. If we go back to Aristotle's *Posterior Analytics*, chapter II, 19, we read:

> Thus sense-perception gives rise to memory, as we hold; and repeated memories of the same thing give rise to experience; because the memories, though numerically many, constitute a single experience. And experience, that is the universal, when established as a whole in the soul – the One that corresponds to the Many, the unity that is identically present in them all – provides the starting-point of art and science: art in the world of process and science in the world of facts.[51]

Spinoza; see Wiep van Bunge, *From Stevin to Spinoza, An Essay on Philosophy of the Seventeenth Century Dutch Republic*, 2016.

48 Simon Stevin, *De la vie civile 1590*. Lyon, ENS éditions, 2005, edited by Catherine Secrétan and Pim den Boer, p. 57.

49 Simon Stevin, *De la vie civile*, pp. 79–80. In this passage, Stevin equates sentiment and beliefs, puts as an axiom the following of our sentiments (the truth is not accessible with certainty, so this is the more practical axiom). This ethics speaks to sixteenth-century mathematicians who were often engaged by municipalities for trials, when they bring, for example, some expertise in heritage problems and commercial or legal disputes.

50 "Parce que l'Arithmetique (ce qui est aussi commun aux autres ars) s'explique par motz comme signes de l'affection de l'ame, lesquels se denotent par escriptures"; Simon Stevin, *L'Arithmetique*, p. 2v, PW IIB, p. 494.

51 This translation is made by Hugh Treddeninck, in the Loeb Classical Library. A description of this passage and its uses in Middle Ages is made in Alain de Libéra, *La querelle des universaux*, Paris, Seuil, 1996, pp. 96–99.

From this passage, we can extract the idea of "induction" and the idea of "varietas".[52] Induction, according to the *Dialectike*, is defined by syllogism:

> Definition XLV. The inductive syllogism is the one in which we show the thing by many examples from the same material.
>
> Explanation. […] With these syllogisms we prove also the principles. By example, to someone who doesn't know that the part is smaller than the whole, we demonstrate then: This part is smaller than its whole, that part is smaller than its whole, and the same for every other part, Every part is smaller than its whole.[53]

The example Stevin chose to exemplify induction is interesting for bringing to light the articulation between dialectic and mathematics. The induction is classically, in Aristotle, a rhetorical technique; the example chosen here by Stevin shows though that the Flemish ingénieur belongs to the tradition of Rudolph Agricola, Erasmus, Philip Melanchthon or Pierre de la Ramée. Rhetoric and dialectic are intertwined; both are needed to produce knowledge, whatever the Aristotelian division between certainty – with syllogisms and axioms – and probability – induction and examples – (*Posterior Analytics* for science and *Topics* for persuasion are then mixed together). The rhetorical induction (a syllogism) is, according to Aristotle's *Posterior Analytics*, beginning from existing practices; it is a way to prove a mathematical axiom and to engrave on the soul the signs that correspond to discourses in the right language (the mother tongue) and to right order.

In Stevin's practice, the idea of an impregnation of the classics or the possibility of an experience of truth realizing the concord between different *auctoritates*:

> [Dialectic] is the foundation of rhetoric. In short, it brings human beings to true beliefs and the number of the *savants*. It's value, not without reason, has been propagated by Stoics (as shown by Laertius in his Philosophers Lives), Plato in Phaedrus, Aristotle in Topics, Cicero, Quintilianus, Boetius and many others.[54]

This truth is acquired patiently by everyday life practices (so the Flemish language is important, and theory – *spiegheling* – is always completed with practice – *daet*). Knowledge is meant to find the good order of presenting these practices[55] ("le bon ordre en la description et instruction des arts" of the "Age of the Sages"); there is frequently in Stevin's texts a difference between the "*ervaren*", those who know, and the

52 Giovanna Cifoletti, "Renaissance: Series of Problems as Varietas", in Alain Bernard (ed.), *Les séries de problèmes, un genre au carrefour des cultures*, EDP Sciences, Labex HASTEC, 2015.

53 "Definitie XLV. Beweghende bewysreden is, inde welcke men door vele exempelen van een selfde materie, de Saecke bethoont. Verclaringhe. […] Door dese bewysreden beproeftmen bequamelick de Beghinselen. By exempel, yemant ontkent alle Deel minder te wesen dan sijn Heel, Men bewijst dat aldus: Dit Deel is minder dan sijn Heel, dat Deel is minder dan sijn Heel, ende also met yeghelick ander Deel, Alle Deel dan is minder dan sijn Heel". Simon Stevin, *Dialectike ofte bewijsconst*, p. 49.

54 "Sy is 't fondament der Rhetoriken; In somme, sy brengt den Mensche tot acht, ende onder 't ghetal der gheleerden: Inder voughen dat de Stoici (als Laertius int leven der Philosophen verhaelt), Plato in Phaedro, Aristoteles in topicis, Cicero, Quintilianus, Boetius, ende vele meer andere, hare weerdicheyt, niet dan met groote reden, soo wijt verbreyden", Dialectike, f3r.

55 See Pascal Dubourg-Glatigny et Hélène Vérin, *Réduire en art. La technologie de la Renaissance aux Lumières*, FMSH, 2008.

students. We can write what we think but we cannot translate our experience to paper. What can be done is for the teacher to demonstrate his experience in clarity and in order. Dialectic, mixed with rhetoric, is an "art of writing", and writing following good order. Stevin chose the "mathematical style". Euclid's *Elements* are the example. The truth is reached by putting in place all the elements allowing the adequation between our words and things, and the good understanding of Nature: Stevin's motto, "wonder en is gheen wonder", the miracle is that there is none in a process of unveilment by codification of knowledge already in the Nature.

As a conclusion of this third part, we can see now the consistency of the "Age of the Sages" theory, and why it is interesting to show the full *mathesis* which lays behind. The question asked by Klein of the definition of number and what is at stake in the Renaissance with algebra can be reviewed in the light of these broader changes.

11.4 Stevin's number

There are four main treatises where we can reach the Stevin's own conception of number. We'll focus on the first two which were published in 1585, *L'Arithmetique* and *De Thiende* (also translated *La Disme* in French in *La Pratique d'Arithmetique*). The other two are *Appendice algebraique* (1594) and *Wiscondighe Gedachtenissen* (1608).

11.4.1 La Disme

Klein emphasizes in his chapter on Stevin the key role of "Arabic digital system". *La Disme* is the small treatise where Stevin explains the use of these decimal fractions. First, he explains what he wants to do without fractions calculus and do only with integers. He then remarks that in a number, for example 1111, each 1 is the tenth part of the precedent, as in a geometrical progression with a reason 10. He uses therefore a particular notation, to go to the right, for the decimal part, as we go to the left, for the integer part, writing (n) as the nth power of 1/10.[56] This notation allows him, for example, the writing of 13 + 1/3 as 13 + 1/3 (0) or 13(0) 3 + 1/3 (1) [13 + 3/10 + 1/30] or 13(0) 3(1) 3 + 1/3 (2) [13 + 3/10 + 3/100 + 1/300]. The Flemish engineer prefers here the choice of precision than that of perfection, here about the number 13 + 1/3:

It is true, that 13 (0) 3 (1) 3 1/3 (2), or 13 (0) 3 (1) 3 (2) 3 1/3 (3) etc. shall be the perfect quotient required. But our invention in this Dime is to work all by whole numbers. For seeing that in any affairs men reckon not of the thousandth part of a mite, grain, etc., as the like is also used of the principal geometricians and astronomers in computations of great consequence, as Ptolemy and Johannes Montaregio, have not described their tables of arcs, chords or sines in extreme perfection (as possibly they might have done by multinomial numbers), because that imperfection (considering the cope and end of those tables) is more convenient than such perfection.[57]

What is important here is the fact that the rules of the four operations given in *La Disme* are those of the numbers inside circles, not the rules of the numbers as decimal fractions. (0) refers *in this case* to 1, (1) to 1/10, (2) to 1/100… but these rules could

56 I use the same technique than Girard to note the Stevin's powers of the unknown (or the tenth…), with parenthesis instead of circles!

57 Simon Stevin, *De Thiende*, Leyden, Christophe Plantin, 1585, p. 20, PW II A, p. 423 for the English translation.

also correspond to (0) as the number of units, (1) as the unknown, (2) as the square of the unknown… In each case, whatever the determination or the indetermination of the number corresponding to the circle, a number of (1) multiplied by another number of (2) will be equal to a number of (3), for example, in virtue of the rule that to multiply (m) and (n), we do (m+n). The rules of calculation with the numbers in circle are more important than the fact that the numbers are determinate or not. In *La Disme*, this general rule is shown on the obvious following example:

But to show why (2) multiplied by (2) gives the product (4), which is the sum of their numbers, also why (4) by (5) produces (9), and why (0) by (3) produces (3), etc., let us take 2/10 and 3/100 which (by the third definition of this Dime) are 2 (1) 3 (2), their product is 6/1000 which value by the said third definition 6 (3); multiplying then (1) by (2), the product is (3), namely a sign compounded of the sum of the numbers of the signs given.[58]

In *L'Arithmetique*, this rule will be used more generally.

We couldn't minimize here the role played by images, explicit visualization and then mathematical notation in printing as a pedagogical tool. (3) can mean the third power of a determinate number or of an indeterminate unknown. Later, Viète, for example, continues to use different denominations: latus, quadratum, cubus, quadrato-quadratum… for numbers; these powers are generalized with longitudo, planum, solidum, plano-planum….[59] This is another difference with Diophantus whose texts were made to be read out loud: this double standard also exists in his *Arithmetica* ("tetragonon" is not "dynamis", for example) and a specific notation only for the indeterminate numbers in Xylander edition of 1575.[60]

11.4.2 **L'Arithmetique** *and its four theses*

In definition II of *L'Arithmetique*, Stevin writes: "Number is that by which the quantity of each thing is revealed" ("Nombre est cela par lequel s'explique la quantité de chacune chose")[61] and gives a syllogism to explain why one is a number (this is his first thesis), contrary to the classical definition of number, which considers unit as a *principium*, an *archè* of numbers:

> The part is of the same matter as its whole, / Unity is part of a multitude of unities, / Hence unity is of the same matter as the multitude of unities; / But the matter of a multitude of unities is number, / Hence the matter of unity is number.
>
> Who denies this behaves like one who denies that a piece of bread is bread. We can also say:
>
> If we subtract no number from a given number, then the given number remains, / If three is the given number, and if from this we subtract one, which – as you claim – is no number, / Then the given number remains, that is, three remains, which is absurd.[62]

58 PW II A, pp. 416–417, *La Pratique d'Arithmetique*, p. 145.

59 Viète, In artem analyticem Isagoge, Tours, Jamet Metayer, 1591, Caput III.

60 N, Q, C, QQ, QC… Xylander, *Diophanti Alexandrini Rerum Arithmeticarum Libri sex*, Basel, 1575, pp. 1–2.

61 Simon Stevin, *L'Arithmetique*, Plantin, Leyde, 1585, p. 1v. The verb "revealed" is used in Klein's GMTOA translation (p. 191) and not "explained", as in the translation of PW II B, p. 494.

62 Simon Stevin, *L'Arithmetique*, Plantin, Leyde, 1585, pp. 2r-v, PW, II B, pp. 495–496.

The argument of Stevin is *ad absurdum*; if one is not a number, a piece of bread is not made up of bread, using a material example. The important part here is the fact that Stevin identifies in the fourth line the number with a "matter of a multitude of unities". The materiality is therefore more important than the conception of the multitude of unities. If the multitude of unities is the traditional definition of number, thought as whole numbers, the general number in Stevin's thought is more than that. We can see the importance of materiality a first argument of the fact that every quantity can be considered as continuous in Stevin's point of view, that is, an engineer's point of view, focusing on physical and natural objects.

A great part of Stevin's conception of number is related to the fact that number can be with or without dimension. The first are called "arithmetical numbers" (definition VI) and the second are "geometrical numbers" (definition XIV to definition XVIII). The notations introduced in *La Disme*, corresponding to geometrical numbers, will serve the exact same way in algebra; so the manner Stevin defines them at the very beginning of *L'Arithmetique* is very important. These very first definitions constitute the foundations ("fondements"), like the *Elements* in Euclid, and they are verified and justified by their efficiency in practical disciplines.

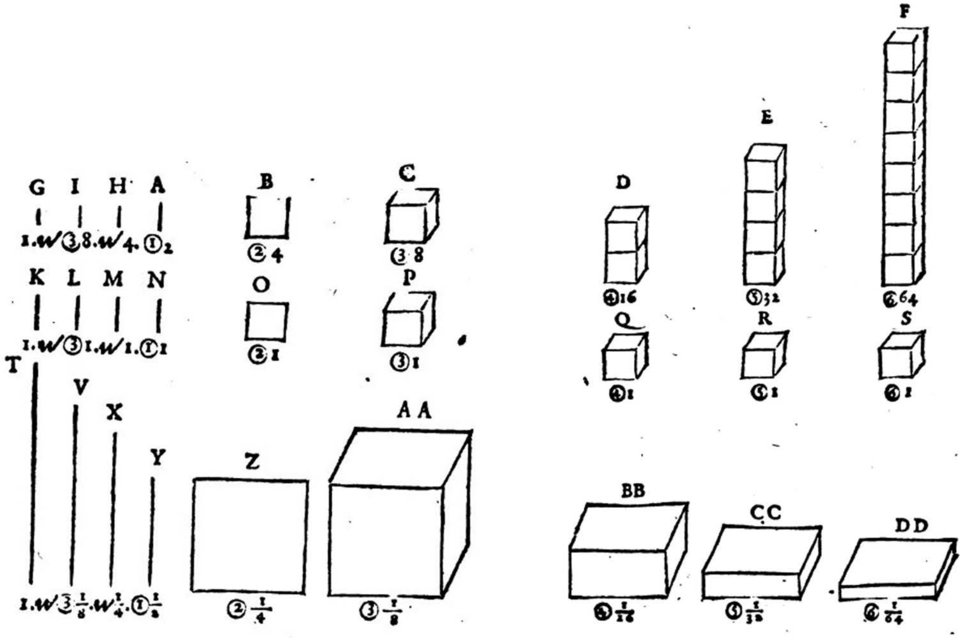
Geometrical numbers are built, thanks to geometrical progression, and each member of this series will be named after its rank. The first will be called "prime" and receive the notation (1), the second "second" (2), etc. They are called geometrical numbers because they can be represented with geometrical figures: the first with a line, the second with a square and so on (should a (0) be represented with a point?). A given number, for example 2, will be a (1), 4 will be a (2) (its square), 8 will be a (3) (its cube), etc. The calculus can be made with the numbers but also with their status in the progression; the (5) (here 32) is the result of the multiplication of the (2) (respectively 4) and the (3) (respectively 8). Knowing that (5) is the result of an operation with the (2) and the (3) is independent of the knowledge of what is the real value of (1), determinate or indeterminate:

After the ancients perceived the virtue of the progression of numbers like 2, 4, 8, 16, 32 etc. or 3, 9, 27, 81, 243, etc. […] they saw it was necessary to give proper names to these numbers, by which we could distinctly signify, calling the first in order "prime", that we will give the sign (1), and the second in the order they named it "second", which we indicate by (2), etc. for example:

(1) 2.	(2) 4.	(3) 8.	(4) 16.	(5) 32.	(6) 64, &c.
		Item			
(1) 3.	(2) 9.	(3) 27.	(4) 81.	(5) 243.	(6) 729, &c.

Then, seeing that the first number was like the side of the square, and the second its square, and the third the cube of the first, etc. and that the similitude of numbers and magnitudes shows several secrets of numbers, they also give them magnitudes' names; they call the first side, the second square, the third cube, etc. and therefore these numbers in general are geometrical numbers. But considering the usefulness of the perfect understanding of the community of these numbers with their magnitudes, we will describe these magnitudes by order with their foundation.[63]

63 "Après que les anciens auoient apperceu la vertu de la progression des nombres ceux ci 2.4.8.16.32. &c. ou 3.9.27.81.243, &c. […] ils ont veu qu'il estoit necessaire, de donner des propres noms à ces



Defining geometrical numbers (see definition XXXI) in this way, we see that every number can be a square number, or a cubic number... We can put every number on any given rank in a geometrical progression. If we put, for example, 2 on the second rank, the first rank (1) will be occupied simply by its square root. In this series, 2 is a square, and square root of 2 is like a classical geometrical number, a "(1)". Therefore, Stevin rejects the terms absurd, irrational..., used sometimes before him. We have here three of the four "mathematical thesis" about numbers Stevin reminds us at the end of *La Pratique d'Arithmetique*.[64]

nombres, par lesquels on les pourroit distinctement signifier, appelans le premier en l'ordre Prime, que nous signifierons par (1), et le deuxiesme en l'ordre ils le nommoient Seconde, que nous denoterons par (2), & ainsi des autres, par exemple: (1) 2. (2) 4. (3) 8. (4) 16. (5) 32. (6) 64, &c. Item (1) 3. (2) 9. (3) 27. (4) 81. (5) 243. (6) 729, &c. Puis voians que ce premier nombre, estoit comme costé de quarré, & le second son quarré, & le troisiesme le cube du premier, &c. & que ceste similitude de nombres & grandeurs, manifestoit plusieurs secrets des nombres, ils leur ont aussi attribué les noms des grandeurs, appellans le premier Costé, le second Quarré, le troisiesme Cube, &c. & conséquemment tous ces nombres en general Nombres Geometriques. Mais consideré l'vtilité de la parfaicte intelligence de la communauté de ces nombres auec leurs grandeurs, nous descriprons ces grandeurs par ordre comme leur fondament". Simon Stevin, *L'Arithmetique*, pp. 9–14.

64 To see each number as an algebraical/cossist number, following geometrical progression is to found in Stifel's *Arithmetica integra* (1544). Via another cossist book, Scheybl's *Algebrae compendiosa facilisque descriptio qua depromuntur magna arithmeticas miracula* (1551), these comparisons can be found in French mathematics, particularly Pierre Forcadel, L'Arithmetique (1556). This circulation is important to compare Stevin and Viète. See Giovanna Cifoletti, *Mathematics and Rhetoric: Jacques Peletier, Guillaume Gosselin and the French Algebraic Tradition*, PhD thesis, Princeton, 1992. The second part of François Loget's article, "L'algèbre en France au XVIe siècle: individus et réseau", in

By generalization, algebraic numbers are also quantities (definition XIX), 3/4 (1) for example, and an "algebraic multinomial is a number composed with different quantities" (definition XXVI), 3 (3) + 5 (2) – 4 (1) + 6 is the given example. After Stifel and the cossist tradition where algebra completes arithmetic, the algebraic number is the most sophisticated description of numbers in general, following a progress begun with integers and fractions.

Stevin built his definitions to do so, and proceeds by analogy. To show two examples among many, he has defined the two signs + and – in order to separate two numbers which are incommensurable. Then, when Stevin solves the problem of the addition of two algebraic numbers, he explains that 2 (1) and 3 (2) are incommensurable and their sum is therefore 2 (1) + 3 (2). As a demonstration, he just says: "The demonstration of aforementioned examples is obvious by the demonstrations of problems of previous additions".[65] He refers to the 28th problem, the one solving the sum of radical multinomials. Stevin considers the addition in algebraic numbers to be the same as previous additions. He has defined algebra as the rule of false position with algebraic numbers and when he comes to algebra in the last and the 81st problem, he says "we have declared the method by analogy; with arithmetical numbers in the 16th problem [rule of false position with arithmetical numbers], we will show it now by its effects".[66]

This generalization raises a limit to Klein's work: the importance of geometric progression shows the importance of the operations, and not only the definition of what is a number.

The passage from arithmetical number to geometrical number, the analogy with the bread, the engineer's point of view or the four theses imply that, according to Stevin, "number is not a discontinuous quantity".[67] However, his demonstration is less convincing. There are two pieces to put together here. The first one about the convention of what is a unit; the second is a discussion about 0 and 1.

To give proof of what Klein said about the change of the unit of measurement to consider modern numbers not as a multitude of units,[68] Stevin adds after the demonstration of the continuity of numbers:

> When you deal with division in your imagination of this unique and whole quantity proposed in 60 unities (what you could also divide in 30 dualities, or 20 trinities, etc.) and then after that you define what is divided, this is not the definition of what is dealt here: you could similarly divide in your imagination the proposed magnitude in sixty parts, and then by the same reason define each part as being a discontinuous quantity, which is absurd. As is the general communion between

Sabine Rommevaux, Maryvonne Spiesser, Maria Rosa Massa Esteve (ed.), *Pluralité de l'algèbre à la Renaissance*, CESR, Honoré Champion, 2012, pp. 69–101, explains Forcadel's views on number, which are close to Stevin's.

The four theses are the following: (1) the unit is a number; (2) Every number can be a square, a cube…; (3) Every root is a number; (4) There is no absurd, irrational… numbers.

65 Simon Stevin, *L'Arithmetique*, Plantin, Leyde, 1585, p. 238.

66 Simon Stevin, *L'Arithmetique*, Plantin, Leyde, 1585, p. 398.

67 Simon Stevin, *L'Arithmetique*, Plantin, Leyde, 1585, p. 4v.

68 In the quotation of Klein in my introduction: "Nothing now stands in the way of changing the unit of measurement in the course of the calculation and of transforming all the fractional parts of the original unit into "whole" numbers consisting of the new units of measurement".

> magnitude and number to the parts, the same applies here; that is, to know, to the continuous magnitude corresponds the continuous number which we attribute to it and such discontinuity received the magnitude after a division; the same discontinuity is said of the number. To speak of an example, number is something in magnitude, as humidity in water, for this is extended in every and each part of the water. Similarly, number for magnitude is extended in every and each part of the magnitude. […] In this way, these two quantities can't be distinguished by continuity or discontinuity.[69]

There is then an arbitrariness of the unit, to measure a line or a number. Beyond the definitions of number and the commonalities between arithmetical numbers and geometrical magnitudes, this gives to the operations a great importance. These operations must apply to all of these magnitudes.

The geometrical point is related to zero; a line can represent every other number. A line is not longer if we add to it a point, as a number is not greater if we add zero to it. In geometry, a line AB can be extended until a point C, so that AC is a continuous line. Stevin explains that the correspondence of this operation in arithmetic consists of considering a line from a point D, corresponding to the number 6 to a point E corresponding to the number 0, so that DE is 60. Therefore, the way we write 60 with 6 and 0 corresponds to numbers as continuous quantities.[70] Perhaps Stevin wanted to relate this geometry to the Arabic digital system. We can quote indeed on the same problem a passage from the "Age of the Sages" theory about arithmetic. Stevin comes back to the tenth, to justify the double relation 0 / point and 1 / line:

> These numerals of the tenth progression have come to light again from Arabic books, so that, beginning with this, we can learn what is the origin or point of number which the Age of the Ignorant (I mean from the beginning of the famous Greeks up to the present), misunderstanding of it stated to be unity.
>
> For the Noble and Very Learned Mr. *Josephus Scaliger* has shown me that the Arabs drew a point for it, as follows: also calling it point, and these points were marked underneath the numerals where we put 0 which corresponds to what we said about it some years ago in our French Arithmétique, in the 2nd definition.[71] I consider that the reason why 0 is now used instead by Europeans is that we are accustomed to use points at the end and the demarcation of written sentences in which points also often succeed numbers; if points were used there it would raise doubt as to whether it was a point belonging to the number, increasing the latter unduly, or a point destined to demarcate the sentence; and in order to prevent such doubt, the point was changed and replaced by a 0. Now then, since in the Age of the Sages 0 was called a point, in order to imitate them we shall henceforth also give it this name and call it Numerical Point so as to distinguish it from the geometrical point; we are abandoning the previous name of *Commencement*, which we hitherto used for it. As to the fact that some people who aren't able to judge sufficiently by means of natural reason, act on the authority of those who

69 Simon Stevin, *L'Arithmetique*, Plantin, Leyde, 1585, pp. 5r–5v, PW, II B, pp. 502–503.
70 Simon Stevin, *L'Arithmetique*, Plantin, Leyde, 1585, pp. 3v–4r, PW, II B, pp. 499–500.
71 This is the exact passage we try to clarify.

> have dealt with the subject, I consider that they are on the right road, provided they follow the most authoritative authority. In other words, i.e. the skilled arithmeticians of the Age of the Sages, did not follow the Greeks, were not arithmeticians nor could perfectly be so, through lack of the proper numerals, as has been said above.[72]

So Stevin seems to reject what he did in *La Disme*, where he defined as a "commencement" (in flemish, *beghinsel*, which can mean "beginning", "principle" or "element")[73] not only 0, but every integer seen without dimension: "For example, some proposed number like 364, we call it 364 commencements, writing it like this: 364 (0)". By analogy, a "commencement de quantité"[74] (definition XIV) will be the "constant" in a multinomial expression, because (0), (1),… are in this context the powers of a determinate or indeterminate number.

11.4.3 Klein's interpretation

Klein continues his long review of Stevin's "wise age" by emphasizing the decimal system found in *La Disme*,[75] developed from Arabic arithmetic which allows the introduction of a new concept of number disconnected from the ancient "arithmos" (numbered assemblage):

> Now it is characteristic of his bias that he always draws invidious comparisons between Arabic and Greek science. The basic reason for this is the Arabic digital and positional system which he is inclined to view as the heritage of the "wise age" since it appears to him to be immeasurably superior to the Greek notation. On the basis of this Arabic digital system he undertakes a fundamental critique of the traditional concept of "numbered assemblage" (arithmos), beginning with the concept, crucial indeed, of the "one," the "monad," the "unitas."[76]

It is clear that the possibility of the division of the unity (Stevin is following here Diophantus IV, 33; V, 12, 13, 14, 15),[77] and the notation made to write with integers some fractions like 1/3 doesn't fit with Greek notation. The choice of 0 and not 1 as a "commencement" also challenges the concept of number.

These choices of the definition of number disconnect the notation of the number from the definition of number as "arithmos". As we have seen in *La Disme*, and to say it with Klein, "Stevin no longer deals with the number of units which are determinate

72 PW III, pp. 600–601.

73 Christophe Plantin, in his thesaurus Flemish – French – Latin of 1573, proposes several equivalences: Begin/beginsel can be translated in French with the word "commencement", and in Latin with the words "exordium", "primordium", "principium", "elementum" (like fire, earth, air and water, or "het beginsel der letteren", "elementum literarum, A b c") or even "origine" ("het beginsel der werelt", "ab origine mundi"). Stevin wrote in 1586 in *De beginselen der weeghconst* the elements of statics. It's a Euclidian work.

74 Simon Stevin, *L'Arithmetique*, Plantin, Leyde, 1585, p. 15.

75 Simon Stevin, *De thiende*, Christophe Plantin, Leyden, 1585, translated into French "La Disme" in *La Pratique d'Arithmetique*, Christophe Plantin, Leyden, 1585; PW, II, A, p. 389.

76 Klein, GMTOA, p. 190.

77 Stevin, *L'Arithmetique*, pp. 3r–3v.

in each case but with the unlimited possibility of combining ciphers according to definite 'rules of calculation'".[78] It matches the theory of number with the use of calculation which is important for the existence of a theoretical logistics. The convenience and the precision for astronomical tables Stevin spoke about in *La Disme* are also the theme of *L'Appendice Algebraique* of 1594, where equations are not solved with exact formulas, but with more precise approximations; this happened perhaps under the influence of Archimedes Stevin which studied in 1586.[79] Stevin is definitively modern, as Diophantus.

Klein develops, in the core of his chapter, a discussion of the syllogism explaining why one is a number. This syllogism reveals the symbolism of Stevin's arithmetic:

> The decisive premise is the one in which the "material of a multitude of units" is equated with "number." Stevin here simply accepts the classical definition of *numerus* as "a multitude consisting of units", but he understands this conceptual determination itself as the "material" of the thing to be defined, in the same sense in which one speaks of the material (materia) of water or of bread. [For just as that which is the "stuff" of bread is understood by Stevin as identical with "what it is," namely bread, so the object intended by a concept is understood by him as a "piece" of the conceptual content taken as a "stuff," a material, which is, in turn, identical with the concept itself, namely with "what the object is."] Only on the basis of such an interpretation is the first premise of the syllogism, according to which the "part" is of the same material as the "whole" relevant. This does not mean that Stevin commits a** paralogism. The fundamental *presupposition* which underlies his understanding — although he hardly recognizes it as such — is precisely the identification of the mode of being of the object with the mode of being of the *concept* related to the object. This means that the one immense difficulty within ancient ontology, namely, to determine the relation between the "being" of the object itself and the "being" of the object in thought, is here (and elsewhere) accorded a "matter-of-course" solution whose presuppositions and the extent of whose significance are simply bypassed in the discussion. The *consequence* of this solution is the *symbolic* understanding of the object intended, an understanding in which its *actual* objectivity is posited as identical with the mode of being of a "general object," or, in other words, in which the object of an "*intentio secunda*" (second intention), namely the concept as such, is turned into the object of an "*intentio prima*" (first intention). Such a symbolic understanding of *numerus* or *quantitas* is precisely what is *presupposed* in this syllogism.[80]

The passage about materiality is important because Klein considered the identification of the general object of *mathesis* with the substance of the world as "Descartes' great idea".[81] The falsity of ancient definition indicates that Klein, following Stevin's argument, is the "absence of the necessary equipment, namely of ciphers", particularly the misuse of the point and the lack of a sign for nought. As we have seen here, Stevin builds a parallel between arithmetic and geometry; 0 is for a number what a

78 Klein, GMTOA, p. 193.
79 PW II B, p. 740.
80 Klein, GMTOA, pp. 191–192.
81 Klein, GMTOA, p. 197.

point is for a line, and "0" is in 1605 in the *Wiscontighe Ghedachtenissen*, following the *Age of the Sages* theory, called "poinct de nombre" (number point), putting aside a previous word "commencement" (beginning) used in *L'Arithmetique* or *La Disme*. The symbolism lay in the distinction between geometrical numbers with dimension and arithmetical number seen as scalars:

> Definition IV states: "An arithmetical number is one expressed without an adjective of size." (Nombre Arithmetique est celuy qu'on explique sans adjectif de grandeur) In contrast, "roots," "quadratic" numbers, "cubic" numbers, etc., are called "nombres Geometriques", although it should be added "that any arithmetic numbers whatsoever can be squared or cubed numbers, etc." (que nombres quelconques [sc. arithmétiques] peuvent estre Nombres quarrez, cubiques, etc.). But insofar as their "absolute," i.e., numerical, value is not known, the "geometric numbers" enter algebraic computations as indeterminate "quantities" and are designated in the following way: (1) (2) (3) (4) etc. (corresponding to our symbols x, x^2, x^3, x^4, etc.). Now just as 0 is the "beginning" of "arithmetic" numbers, so any "arithmetic" number you please is the "beginning" of these algebraic "quantities" — Def. XIV: "The beginning of quantity is every arithmetic number or any radical whatsoever." (Commencement de quantité, est tout nombre Arithmetique ou radical quelconque.) It is thus designated by (0), insofar as its "absolute" value is not known.[82]

In this system, all quantities are symbolic and can be expressed by a combination of "ciphers"[83] (with a limited set of rules) and cannot be seen as the multitude of units. Stevin assimilates "the concept of 'number' to operations on 'numbers'", assimilates also to count and "knowing how to handle 'ciphers'".[84] As Wallis following Stevin does, the general quantity and therefore the "universal algebra" are reached and "explained rather on arithmetic than geometric principles".[85]

11.4.4 Trivium approach of number

Lorenzo Valla himself, the founder of the new dialectic Stevin uses, explained that one is a number. Refusing false speculations on unit (Aristotle and Euclid), because language is what people do with it, and those who speak in a mother tongue are greater philosophers than scholastics: Valla describes a short story:

> Two women shared twelve hens and one rooster among them. They agreed that one would have the eggs on days when the number laid was even, but that the other would get them when the number was odd. 'Say that sometimes single eggs were laid. To which would the egg go; to neither?' 'No, to the one who was due the odd number of eggs.' Therefore, one egg makes a number. Therefore, foolish

82 Klein, GMTOA, p. 196.
83 Klein, GMTOA, p. 193.
84 Klein, GMTOA, p. 197.
85 Klein, GMTOA, p. 215.

> women sometimes know the meaning of words better than great philosophers. Women put words in use; philosophers plays with them.[86]

Considering this dialectical-rhetorical tradition, Stevin is not the first to state that one is a number. We found elements of truth, when these are expressed in mother tongue, as Stevin explained it in 1586:

> Let them attack valiantly, for what have Reuchlinus, Valla, Erasmus, Barbarus, Picus, Politianus, etc.) achieved, who merely protected Latin,· and likewise the French whose arguments and linguistic material we know well enough? What then shall we achieve, who propagate Dutch (O worthy subject!)? Certainly it not only brings the language on a higher level or advance ourselves, but also other nations, which will then adorn not only their houses and bodies with the artistic products of the Dutch, but also their minds with knowledge, for the arts which other nations cannot express in their own words, will here be thoroughly understood from the elements [beghinselen] by the common man, and through his inborn disposition [ingheboren ghenegentheyt] thereto he will be able to advance it, to the profit of all nations, in quite a different way from what is possible to the others.[87]

We have here the part of the "Age of the Sages" reminded by Klein. The induction built on "axioms"/"beghinselen" can in fact be built on collective common knowledge, according to *Posterior Analytics*' first sentence. We see the same kind of discussion in Stevin's *L'Arithmetique*, the year before, in 1585, about mathematics:

> One must know then human beings seeing they had to speak and had the intelligence of the quantity of things, they named each simple things, one; and when at the same was applied another, they called them together two, and when the proposed simple thing was divided into two equal parts, they named each part half, etc. Then, considering that one, two, three, half, third, etc. were proper nouns for the explanation of the aforesaid quantity, they saw that it was necessary to understand all these species according to a gender (because it was their manner of doing similarly for wheat, barley, oat, they named them in the gender grain; eagle, dove, nightingale, in gender bird), which gender they called Number.[88]

86 Lorenzo Valla, *Laurentii Vallae Romani, Dialecticarum disputationum*, Cologne, Ioannes Gymnicus, 1541, p. 18 (livre 1, chapitre 2, "Sex quae vocantur transcendentia, quam vim habeant, ex quibus res principatum probatur obtinere. Caetera non esse transcendentia"). This passage comes from a discussion Valla began in *Repastinatio dialecticae et philosophiae* (1439) and continues in *Disputationes dialecticae* (1441). This translation is the one in English made by Brian Copenhaver and Charles B. Schmitt in *Renaissance Philosophy*, Oxford University Press, 1992, p. 218. This little story seems to be, with variations, common in Middle Ages, as well as the status of number for one.

87 PW I, p. 93, in the *Uytspraeck* (preface of 1586 *De beghinselen der weeghconst*). We note the presence of kabbalists and hermetists in the list, particularly Reuchlinus.

88 "Il faut donc savoir que les Hommes jadis voyant qu'il leur était métier de parler et avoir intelligence de la quantité des choses, ils nommaient chaque chose simple, un; et quand à la même était appliquée encore une autre, les appelaient ensemble deux, et quand la proposée simple chose était divisée en deux parties égales, ils nommaient chacune partie demi, etc. Puis considérant que un, deux, trois, demi, tiers, etc. étaient noms propres et convenables pour l'explication de ladite quantité, ils ont vu

From such a quotation, we could think of the problem of universals, for which the concepts of *prima intentio* and *secunda intentio* Klein used were made.[89] The question raised by Klein about the constitution of number and its relation with unities receives here a different explanation. Number, the material of multitude of unities is now a gender, a category made to subsume the simple thing "one", a set of two same things called "two", etc. The question is now shifted to the relation of the words with things. Shakespeare here comes in mind:

> 'Tis but thy name that is my enemy; / Thou art thyself, though not a Montague. / What's Montague? it is nor hand, nor foot, / Nor arm, nor face, nor any other part / Belonging to a man. O, be some other name! / What's in a name? that which we call a rose / By any other name would smell as sweet;[90]

This nominalism made words a convention. As Marijke van der Wal has interestingly remarked, the Stevin's definition of number is a definition of a name, not of a thing: "Number is that by which the quantity of each thing is explained".[91] "that", "cela" refer to a noun. But according to Stevin, the two can be considered one for another:

> The definition and what is defined are reciprocal. From this, the definition must be short but sufficient, and understandable, explaining the being of the thing. Without ambiguity and ignorance. We must take every such rule as if they were in our mind, but they are described inside the truth.[92]

Since Stevin bypasses pragmatically this discussion and uses the name (in the thought) for the thing, we can say with Klein about number that there's a symbolical understanding. Such arbitrariness of Stevin's definition of the name, inside language, will be criticized by Arnauld and Nicole in their Port-Royal Logic, a century later.[93]

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qu'il était nécessaire de comprendre toutes ces espèces sous un genre (car telle est leur manière de faire en tous autres semblables comme blé, orge, avoine, ils le nomment en genre Grain; aigle, tourterelle, rossignol, en genre Oiseau) lequel genre ils appelaient Nombre". Simon Stevin, *L'Arithmetique*, Plantin, Leyde, 1585, p. 2v.

89 Alain de Libéra, *La querelle des universaux*, Paris, Seuil, 1996.

90 William Shakespeare, *Romeo and Juliet*, 1585, Act II, Scene II.

91 See Marijke J. Van der Wal, "Logic, Linguistics, and Simon Stevin in the Context of the 16th and 17th Centuries", in Kurt R. Jankowsky (ed.), *Studies in the History of the Language Sciences*, History of Linguistics, 1993, pp. 147–156.

92 "Als dat Definitie, ende Ghedefinierde malcandertreffen: Daerbeneven, dat de Definitie cort, nochtans genouch, oock verstaenlick sy, verclarende het wesen der saecke: Sonder Verkeerdespreuck: niet Ontkennende. Alle welcke Reghelen wy meughen nemen als voor ons Wit, maer inder waerheyt sy sijn beschrijvelicker, dan naervolgelick, want overmidts weynich saecken inde werelt sijn, die niet met velen anderen seer groote gelikckheyt hebben, so ghebeurt het selden, ofte beter gheseyt nimmermeer, datmense so volcommentlick Definieren can, als de voornoemde conditien heysschen". Simon Stevin, *Dialectike*, p. 53.

93 See Marijke J. Van der Wal, op.cit. For a critique of language in the Valla's reform of dialectic, see Luce Giard, "Lorenzo Valla: la langue comme lieu du vrai", *Histoire Épistémologie Langage*, tome 4, fascicule 2, 1982, pp. 5–19; Tristan Dagron, "Plurilinguisme philosophique et crise du concept: le moment humaniste", *Réforme, Humanisme, Renaissance*, n. 64, 2007, pp. 11–29.

11.4.4.1 Material cause

The relation between the "being" of the object itself and the "being" of the object in language can also be seen in Stevin's *Dialectike* about materiality. In the decisive syllogism commented by Klein, the analogy between a number and bread was made. Again, we can start from what it is said in the *Dialectike* to understand the *L'Arithmetique*, and particularly from the definition of the material:

> DEFINITION II. Matter is the cause by which something is made.
> EXPLANATION. The Matter, from the latin word Materia, is called in Dutch "Stoffe", or as we say often (as some people say too) "Stof", derives its meaning (as it seams) from epicureans, who were careful to substantial questions on matter.

After citing the atoms of Epicure,[94] and saying that we say that Astronomy is a deep "matter", Stevin continues:

> Matter, says the definition, is the cause by which something is done, like the matter of velvet is silk, of clothes is flax, of armour is iron. It's the same for non substantial matters (each one is attributed following the substance by analogy): for the arithmetic (as its name indicates), it is the number, for the music, it is the sound, for the dialectic, it is the question or the argumentation.[95]

Number is a material of the non-substantial thing ("materie der onlichamelicker dinghen"), and its properties can be searched by analogy with material of substantial thing. This also helps to see number as a number of something.

By analogy, thanks to mathematics, we can expand the role of mathematics. For example, in the *Elements of the Art of Weighing*, Stevin explains:

94 A difficult articulation between continuity and discontinuity (atoms and continuous matter, discrete and continuous quantities) is, according to the anthropologist Philippe Descola, a characteristic of the analogism, one of the four ontologies which allows us to understand the relation between nature and culture in a society. So we could say that Stevin, definitely a Renaissant scholar, is still on this particular matter, a representant of the medieval analogism, and not a representant of the modern naturalism. Philippe Descola, *Par-delà nature et culture*, Folio essais, 2005, pp. 351–353.

95 "Definitie II. Materie is d'oirsaecke, dat yet afghemaeck wort. Verclaringhe. Materie vanden Latijnschen woorde Materia wort opt Duytsch eyghentlick geseyt Stoffe, als oftmen wilde seggen (ghelijck by sommighe Duytschen oock ghebruyckelick is) Stof, hebbende sijn oirspronck (soot schijnt) uyt de meyninghe der Epicureen, die alle lichamelicke saecken achten van Stof gecomen te zijne, te weten dat Stof, 'twelck over al inde locht drijft, ende best ghesien wort, inder Sonnen raeye door een gat in huys schijnende, gemenelick naer het Griecx Atomus gheheeten, ende en hadde ons de gewoonte, van onse eyghene tale niet ghebrochtaen een vreemde, men soude het woort Stoffe, hier bequamelick meughen gebruycken, want datmen voor, d'Astronomie is een diepe Materie, seyde d'Astronomie is een diepe Stoffe, ten waer niet so oneygen, als ongehoort, maer als voren gheseyt is, wy sullen ons ghevoughen, naer t'gene nu ter tijt best verstaen wort. Materie dan (seght de definitie) is d'oirsaecke daer yet afgemaeckt wort, als de Materie van Fluweel, (overmidts het van zijde gemaeckt wort) is Sijde, van Lijwaet, is Vlas, van een Harnas, is Yser: Alsoo oock der onlichamelicker dinghen Materie (welcke naer de lichamelicke haer ghelijckspreuckelick toegevoucht wort) als der Arithmetiken (want sy daer uyt bestaet) is Ghetal, der Musiken, Geluyt, der Dialectiken, Question ofte Argumenteringe". Simon Stevin, *Dialectike ofte bewijsconst*, Definition II, pp. 4–5.

> If plane figures had any weight, and these were admitted to be proportional to their magnitudes, we might properly speak of their gravity, centre of gravity, centre line of gravity, etc., but since a plane figure has no weight, properly speaking there is no gravity, centre of gravity, nor centre line of gravity therein. Therefore all this has to be understood metaphorically, and it has to be assumed by hypothesis that the weights of plane figures are proportional to their magnitudes, for: THE FALSE IS ADMITTED IN ORDER THAT THE TRUE MAY BE LEARNED THEREFROM.[96]

Therefore, the same mathematics can be applied to substantial or non-substantial things. We could say that symbolism corresponds to his engineer life.

11.5 Conclusion

Klein saw that he had to describe Stevin's epistemology to clarify his demonstration of what is a number according to the Flemish *ingénieur.* Text after text, Stevin built a consistent corpus and thanks to other sixteenth-century debates we can begin a description of his "art of thinking". We have to navigate, from discipline to discipline, though, in order to have the elements allowing us to judge what is at stake in the debate about the status of number and unity.

By reasoning from practices, as Aristotle guided us in *Posterior Analytics*, Stevin wrote book after book of what became a kind of *encyclopaedia*, structured by a logical pillar made of mathematics and dialectic. As in the Stoic epistemology, every part of it (logic, ethics, physics) is closely related to the other and to speak of one implies the understanding of the others.

We have seen that the definition of number is made with dialectical tools fit to our use, particularly the material cause and the unification of trivium and quadrivium, or the analogies made between arithmetic and geometry. From all the twists and turns we have made, it becomes clear that Stevin's number is at the centre of an articulation of many disciplines. We saw that the dialectic Stevin used is influenced by the idea of matching different levels of concepts, as his arithmetic unifies the different kinds of numbers. The fusion of algebraic operations with arithmetical ones (thanks to the vision of algebraic numbers seen as a generalization of geometrical numbers) is the strong continuity between what are mathematical objects of second intention, algebra, and what are mathematical objects of first intention; in addition, a well-defined concept of number suggests that there is symbolic algebra in Stevin. In the end, Klein's great philosophical intuition using medieval intentions to compare number in antiquity and number in Renaissance has also a historical basis.

96 *Elements of the Art of Weighing*, PW I, pp. 224–227, the dialectical vocabulary is also omnipresent in these pages. See also in 1585 *Dialectike* the definition of a metaphor (when we say something false to mean something true), *Dialectike ofte bewijsconst*, Christophe Plantin, 1585, p. 17, definition XII).

12 “New” early modern evidence for Jacob Klein’s theses

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Abstract: We start from the question: what kind of quantity were the quantities algebra talked about, according to sixteenth-century algebraists? The question springs from Klein’s theses, which we would formulate as follows: Greek numbers were collections of units. Unit was not a number but that by virtue of which anything is called one, as Euclid and Aristotle said. Diophantus introduced abbreviations for the unknown quantities but only Viète’s algebra (together with Stevin’s and Descartes’) produced the shift to symbol. Hence, in early modern Europe, these quantities came to be conceived as *secundae intentiones* in a shift of *Begrifflichkeit* (conceptuality) from Greek numbers. Klein also states that this shift in conceptuality depended on the apprehension of the mathematical past by these mathematicians. But what happened in detail in sixteenth-century algebra with respect to this transformation; what appeared before Viète’s *species*? First, I examine Niccolò Tartaglia’s philosophical discussion about unit, number and some crucial notion Ibn Rushd Tartaglia was familiar with. Tartaglia also ascribes to Savonarola some of his statements, indicating a logical interpretation, leading to *species* and *secundae intentiones*. His mathematical production culminating in the *General Trattato* (1557–1560) develops in great detail many parts of mathematics as listed by Al Farabi’s *On the Sciences* and Gundisalvi’s *Division of sciences*, recently transmitted by Savonarola and including algebra. Going back to the main treatise comparable to Tartaglia’s Luca Pacioli’s *Summa* (1494) and looking at its notion of quantity, we find again Gundisalvi, who has transmitted a few crucial themes relevant to the idea of unit, number, matter and continuous quantity. Finally, a discussion of the solutions for second degree equations introduces sixteenth-century algebraic quantities as a new version of Euclid’s plane and solid numbers, but geometrically constructed from Pacioli to Nunes. Both Pacioli and Tartaglia reconstructed algebra on Euclidean foundations on the philosophical assumption of a primitive doctrine prior to arithmetic and geometry, called *practica speculativa*, speculative practice.
Keywords: Jacob Klein; François Viète; Euclid; Number; Concept of Number; Symbolic Algebra; Mathematical Analysis; Algebra; Al Farabi; Domingo Gundisalvi; Luca Pacioli; Niccolò Tartaglia

12.1 Introduction

The previous papers have addressed the philosophical questions raised by Jacob Klein.

Here, we shall try to participate in the debate, within a larger circle of Klein’s readers, about his actual contributions to the history of mathematics by adding some

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historical elements which have only recently become clear. Anhistorically, we could consider them as missed historical evidence for Klein's theses, but also as a way to situate Jacob Klein's theses in his time.

There are aspects of early modern algebra that Klein did not plan to consider in depth in his famous articles: for instance, among the mathematicians, he barely mentions Tartaglia. With respect to mathematical traditions, Klein did not develop in detail any argument about the Middle Ages. This would include first of all Arabic mathematicians (a particularly crucial point for a history of algebra) but also Byzantine and Hebrew mathematicians. During the Middle Ages, mathematics was not disjoint from philosophy as we can conceive of it, as it was not disjoint from many other sciences. For instance, while Klein was aware at least to some extent (the extent of historical knowledge of his times) of the impact they had on the process – he mentions, for instance, the impact of Arabic numerals in the birth of algebra – he could not predict the discoveries of later historiography in explaining that same history. In consequence, he could not predict to what extent they could confirm or refute his theses, or how they would fit into his rational or phenomenological reconstruction.

I believe that by applying Klein's method – not so much as a grand narrative, difficult for historians to accept nowadays, but as the attempt to reconstruct the "ideal library" of mathematical innovators such as Viète, Stevin and Descartes – we can actually find that some of these neglected aspects are arguments in favor of Klein's theses. Here, we shall deal with some texts by Niccolò Tartaglia (1499–1557), by Luca Pacioli (1447–1517) and by Pedro Nunes (1502–1578), all three basic readings for late sixteenth-century algebraists.

Furthermore, concerning the current strongest criticisms of Klein's historical theses, such as Jeffrey Oaks' article on Viète,[1] I hope to give some elements allowing to identify other ways, more contemporary to Viète, of conceiving quantities.

I shall look at the genealogy of the foundations of algebra as some mathematicians expressed it in early modern times. Inevitably, this genealogy will have something to do with some aspects of *geometric algebra*, the famous controversy of twentieth-century historians of mathematics. A recent reconstruction of the whole history of this debate appears in Jens Hoyrup.[2] The reference is to the second book of Euclid's *Elements*, where we find a calculation (four operations) on segments and quadratic expansions. This was interpreted in the Middle Ages as a calculation on indeterminate quantities in the sense that the identities hold for numbers and for numbers assigned to segments. The origin of this calculation is the theory of the application of areas, due to the Pythagoreans. Many historians, such as M. Cantor, P. Tannery, H. G. Zeuthen and T. Heath, have noticed the special features of this book. Progressively, some of these authors, very aware of recent algebraic discoveries, introduced the idea that the Greeks had algebra but they did not accept it for philosophical reasons. They called it geometric algebra, a historiographical idea very active among historians of mathematics for more than a century. A main actor in this debate was the historian of Babylonian algebra Otto Neugebauer, a mathematician who had studied algebra in Göttingen with Emily Noether and was close to Jacob Klein. According to Neugebauer, the Greeks actually took

1 Jeffrey A. Oaks, "François Viète Revolution in Algebra." *Arch. Hist. Exact Sci.* 72 (2018): 245–302.
2 See Jens Hoyrup, https://www.aimspress.com/article/10.3934/Math.2017.1.128

algebra from the Babylonians, while Van der Waerden went further and accepted the historiographical tradition of the Greeks (Proclus (412–485) and Eudemus (370–300 B.C.) saying that in fact the Pythagoreans themselves had transmitted it through the theory of the application of areas. The story was that after the discovery of irrational numbers, the Pythagoreans developed not only the theory of ratios and proportions, but also that of the application of areas. For both Neugebauer and Van der Waerden, application of areas was algebra in geometrical guise. A long-standing debate has been about whether an algebraic doctrine was in fact hidden in the theory of the application of areas or if this theory was genuinely geometrical, even though useful for calculation, of which algebra was a late consequence. A first phase of this debate culminated precisely with the essays by Jacob Klein we are dealing with, entitled *Greek mathematical thought and the origin of algebra*, published in German in 1934 and 1936. Although they appeared in Neugebauer's journal, these essays contain a philosophical analysis of algebraic thinking that stressed a *rupture épistémologique*, a radical discontinuity between Greek and modern mathematics. Klein went beyond the early modern nostalgia of golden age Greece and revealed that early modern people had a radically different way of conceiving number and indeterminate quantities with respect to their classical heroes.

What I want to stress here is that Renaissance mathematicians were the actual Western inventors of *geometric algebra* as an ancient Greek discovery: they elaborated upon some Medieval traditions concerning the existence of Greek algebra. The main evidence of Greek algebra, according to these traditions, was the application of areas theory presented in Book II of the *Elements*. Sixteenth-century mathematicians considered this book to be the trace of a lost body of knowledge making explicit the art of discovery, and hidden for philosophical reasons or even because it was a very powerful and precious tool. Already in the course of the Middle Ages, some arithmetical and algebraic authors had focused on Euclid's common notions and Book II of the *Elements* as part of their treatment of algebra taken from Abu Kamil. This is the case of Bar Hiyya (Savasorda), in Barcelona and his collaborator and translator into Latin Plato of Tivoli. Leo Corry has already stressed their role and the differences between the Hebrew and the Latin versions of Savasorda's book, that is, the *Liber Embadorum*, produced in 1145. Around the same year in Toledo, a group of scholars wrote the *Liber Mahamelet*, a compilation of what will become abacus mathematics containing the same interpretation of Book II. This computation with segments, purely geometrically conceived, in particular the first ten propositions corresponding to the quadratic expansions or notable products, became associated with arithmetic. Leonardo Fibonacci, in his *Liber Abaci* (1202), explicitly connected Book II with the *Ars rei et census*, that is algebra. By the time Campanus of Novara (1220–1296) wrote his translation, the Euclidean arithmetical books, in particular Books VII, VIII and IX, were transformed by this interpretation of Book II. In fact, as historians have pointed out, Campanus text is not so much a translation, but a commentary. Its sources include mathematical Arabic and Latin texts until the twelfth century. In particular, the identification of the first theorems of Book II and the corresponding arithmetical identities comes from Jordanus Nemorarius' (1225–1260) *Arithmetic*.[3] Other

3 See Sabine Rommevaux, "La proportionnalité numérique dans le livre VII des *Eléments* de Campanus" *Revue d'histoire des mathématiques* 5 (1999): 83–126.

translations kept the original geometrical sense for Book II, and a representation by segments of the arithmetical books, but Campanus' version was the best known, not only in manuscript, but also especially present in earlier printed editions. Later, some new philological findings such as Proclus' *Commentary on the first book of Euclid's Elements* justified this belief in the sixteenth century, because a few ancient texts mentioned a general method for solving mathematical problems. I found more examples of the belief in this primitive algebra among sixteenth-century mathematicians. It was widespread because it was a way to justify a practical, basic mathematical art, a sort of generalization of the rule of three, as a full mathematical discipline, or even an art of thinking originating in classical or mythical times[4] susceptible to be applied to any sort of problems.

In this paper, I shall try to give more elements of this original mathematical theory taken as the mythical beginning of algebra, as an example of how Klein's theses can still be confirmed by new evidence in the history of sixteenth-century algebra. For now, I refer to the vast secondary literature[5] with respect to the most recent phase of the debate on geometric algebra. I take the rich essay by Leo Corry[6] as my starting point and assume that there is a geometrical algebra historically attested in the Middle Ages, and not in the Greek world. Instead, after its early Medieval beginning, this theory took shape in the Western World at the time of translations of Greek and Arabic texts from Arabic to Latin. This underlying theory of geometric algebra developed substantially in the late fifteenth and sixteenth centuries; it appears to be active even in later mathematical texts, such as the one written by some readers of Viète and Descartes.

For early modern mathematicians, to talk about the sense of algebra meant to stress the role of algebra in connecting arithmetic and geometry[7] and giving a foundation to algebra. Leo Corry has presented a parallel path, starting from Campanus' edition of Euclid's *Elements*. Here, I shall start from Niccolò Tartaglia, who assumed the task of explaining in detail his mathematical ideas not only in the body of mathematical material included in his huge summa (*General Trattato di Numeri e Misure*, six volumes in folio) but also quite explicitly in its philosophical introduction.

4 See also my "The Creation of the History of Algebra in the Sixteenth Century," in C. Goldstein, J. Gray et J. Ritter (éds.), *L'Europe mathématique, mythes, histoires, identités*. Paris, Editions de la Maison des sciences de l'homme, 1996, pp. 121–142.

5 See especially Sabetai Unguru, "On the Need to Rewrite the History of Exact Sciences." *Arch. for Hist. of Exact Sci.* 15, 67–114, 1975; Unguru, S., & Rowe, D. E., "Does the Quadratic Equation Have Greek Roots? A Study of 'Geometric Algebra', 'Application of Areas', and Related Problems." *Libertas Mathematica* 1 (1981) 1–49; 2 (1982), 1–62. For a recent account, see the already cited Jens Hoyrup 2017.

6 Leo Corry. "Geometry and Arithmetic in the Medieval Traditions of Euclid's Elements: A View from Book II." *Arch. Hist. Exact Sci.* 67 (2013): 637–705.

7 Many historians of mathematics have contributed to clarify the relationship between number and magnitude in early modern mathematical texts, for instance, Michael S. Mahoney, "The Beginnings of the Algebraic Thought in the Seventeenth-century," in S. Gaukroger (ed.), *Descartes: Philosophy, Mathematics and Physics*, N. Y. 1980; Antoni Malet, "Renaissance Notions of Number and Magnitude." *Historia Mathematica* 33 (2006): 63–81; Jens Hoyrup, *Selected Essays in Pre and Early-Modern Mathematical Practice*. Springer 2019, as well as Jeffrey Oaks; see above.

12.2 Niccolò Tartaglia

After distinguishing rapidly between the different sorts of arithmetic (theoretical and practical), Tartaglia gives a long definition of unit (*Diffinitione della Unità*).

He writes:

> The unit, as defines Euclid in the first definition of the seventh book, is that from which each thing is said to be one. That is, as each animal is said to be an animal from the soul, so each material thing, which is said to be one (masculine or feminine), is said so from the unit. And people appreciate so much this name of one in the nature of things that they call one not only a single man, or a single horse, a single plant, a single stone, or a single ducat, or other money, and other single things (from such a unit) but also those which are many formally are said to be one. This is evident: two material things often are called one pair, ten are said to be one ten <*decina*>, twelve are said to be one dozen, a hundred things one hundred <*centonaro*> a thousand things one thousand <*mearo*>. Similarly a multitude of soldiers, a team, or an army, or one multitude of cattle, or one herd and so on talking about all material things. Furthermore, not only many things are said to be one, but also the part of a single thing is said to be one, because it is clear that the half of a material thing is said to be one half, and so one third of a material thing is said to be one third. Similarly, one fourth, one fifth, one sixth, one seventh, one eighth and so forth. From this it follows that anything which is in the nature of things, is either one or more than one. And there is no thing that can be less than one, because the less than one is nothing.

This description of the one appears classical, founded on the authority of Euclid and Aristotle. However, we have to be careful to understand which Aristotle we are talking about: it appears from the beginning to be a Neo-Platonic and Arabic interpretation. For example, the "number of" constitutes a long list which, while it is here particularly long and elaborate, can be seen in a shorter version in some Medieval treatises, originating in Al Farabi's (872–950) classification,[8] which will be discussed more later. Similarly, the analogy of the unit with the soul, that is the identification of the unit with the form, has clear Medieval references which will be developed later. Tartaglia, who had published the first translation of Euclid's *Elements* into Italian,[9] had commented directly on the first definition of unit in Book VII: unit is the principle of all numbers and in fact of all things. Tartaglia continues

> Many healthy minds without other disposition will understand the previous definition and disposition. Yet I understand that there will be others who will become very confused, and there will be yet many others who (following the Natural consideration), will take for granted that the said unit is any of the ones mentioned above which is called one. I have been of this opinion myself, as it appears clearly in the disposition on unit done in my version of Euclid.

8 See Alain Galonnier, *Le De scientiis Alfarabii de Gérard de Crémone*. 2016, pp. 226–227.

9 *Euclide Megarense philosopho: solo introdutore delle scientie mathematice*. Venice, Venturino Roffinelli, 1543.

It is true that Tartaglia, in his Euclid, lists all these meanings of unit and the one: *Unit is anything that is said to be one…It is the seed of all numbers … it is the cause of measure* (f. 129 v). And he continues by saying what we have already seen in the *Trattato*, that everything wants to be called one.

> 'In all numerable things it will be that many small things together produce a big thing, as speaking naturally twelve pence make a pound <*dodici dinari fanno un soldo*>.

In the *General Trattato*, he introduces a distinction he found in the *De Caelo*. Tartaglia writes:

> Hence in order to enlighten as much as I can the minds of each of them we have to remember that there are two main considerations of it <the unit>, one of the Natural and the other of the Mathematician. The Natural considers things according to being, as according to reason in conjunction with some sensible matter, and all this is stated by Aristotle and similarly by the Commentator in the sixth book of the *Metaphysics*, second commentary, and similarly by friar Hieronimo Savonarola in his philosophy, in the book where he deals with the division of all the sciences.

Here, Tartaglia explicitly mentions as authorities Aristotle, with reference to the *Metaphysics*, "The Commentator" Ibn Rushd or Averroes (1126–1198), with reference to his commentary on the *Metaphysics* and Girolamo Savonarola (1452–1498), with reference to his *Division of the sciences*.

While the definition of the unit is actually Aristotle's, the distinction between the standpoint of the Natural and the standpoint of the mathematician can be found in Averroes' commentaries on Aristotle and in Savonarola's *Division of the Sciences*, as we shall see.

Tartaglia writes:

> Hence the unit according to this consideration will be each of those material things which are called one. But when the said Natural names one of these units he will always name it together with that sensible matter. That he will name it with its material subject: he will say one golden ducat, or one *scudo*, one florin, one pound, one penny, one dinar, one arm's length of cloth, or one pound of silk, or one bezant, or one ounce of saffron, or one carat of musk. And similarly in geometrical measures, he will say one Perch, one step, one Foot, one ounce, and so in astronomical measures, saying one degree, one minute, one second. And similarly, in their parts, he will say half of one arm's length of cloth, one third of a ducat, one fourth of an ounce of gold, and so forth on all the other material things occurring in the art of trading and commerce and in the others. Conveniently, we can call these species of unit natural or denominate units, and each of these is infinitely divisible as to the quantity of its own material subject. The Mathematician also considers things according to their being in conjunction with such sensible matter (as the Natural does also). However, he considers them as abstracted from such sensible matter according to reason, and all this Aristotle states also, as well as the Commentator, in the above mentioned sixth book of the *Metaphysics*, second commentary, and similarly the above mentioned friar

> Hieronimo Savonarola in the above mentioned passage. However the unit according to this mathematical consideration would be a certain indivisible according to quantity (as Aristotle states in the first book of *Posterior Analytics*, fifth text). This unit comes to be almost like the geometrical point, which is also indivisible according to quantity. And there is no other difference between the one and the other except this one, that the Point has a position, that is a site in the line, and the Unit has no position or determined site, and this is stated by Aristotle in the fifth book of *Metaphysics*, twelfth text.

And later,

> Comparison between the consideration of the Natural and of the Mathematicians about the unit, and the difference of them.
>
> In order to obtain that any kind of people can learn and understand better the difference between these two sorts of considerations about the unit, that is of the Natural and of the Mathematician, let me posit this case. There are two men, who take into consideration the same animal, and I make the hypothesis that one of the two men takes into consideration only the body of such an animal, and the other considers only the soul of such an animal. Now I say that the consideration of the first is similar to the consideration of the Natural, and that of the second is similar to the consideration of the Mathematician. And given that the body of such an animal is sensible matter, and divisible according to quantity, we shall say that the body is similar to the Natural unit. Similarly, given that the soul of such an Animal is an insensible thing, and indivisible, we shall say that it is similar to the Mathematical unit. Charles de Bovelles says that this mathematical unit for many reasons should be compared to God, and for this cause I maintain that our ancient wise men attribute this name of unit to the Great Architect.

Notice that this analogy between mathematical consideration and consideration of the soul is not a simple parallel, but, as stressed by Tartaglia himself with the reference to Charles de Bovelles (1479–1567), is based on a theological interpretation of nature as Creation, ordered by God. Furthermore, the definition of number as the result of the process of abstraction from matter is compatible with the notion of soul in Aristotle, who discusses the identification of the soul with number in the *De Anima*.

Going back to Tartaglia's interpretation of the unit, starting from Aristotle's classical definition, we notice that we arrived at a notion of unit that not only is divisible, but infinitely divisible. Jacob Klein has developed some crucial aspects of the classical notion of unit and of its transformations, but he did not directly include this kind of thinking about the unit and the shift it implies. What Tartaglia stresses here is that the Natural sees the unit as a physical unit, as a concrete thing, as something ("the body is similar to the natural unit"). He explains that for the Natural, that is, for the mathematical practitioner of mixed mathematical sciences, the unit is a unit of measure, so that its "substance," like its matter, is divisible: in measure, a pound is divisible into ounces.

Tartaglia also gives a long definition of Number:

> Number, as defined by Euclid in the second definition of the Seventh Book is nothing but a multitude of units. But we should advise that about number there are the same two sorts of considerations as we have said about the Unit, one

according to the Natural, the other according to the Mathematician. The Natural considers Number both according to reason and according to being, joined to those numbered sensible matters, that is according to that material subject, of those natural units composing that Number. This is why he always mentions and names such number together with the material subject mentioned above. He says so many *ducati*, or so many golden *scudi*, or so many florins, or so many pounds, or *dinari*, or *bagatini*, or so many *grossi*, or *pizzoli*, or so many pounds or ounces of sugar, or of cinnamon, or of ginger, or similar matters. He might say also so many Marche, ounces, quarters or carati of gold, or silver, or so many *stara*, quarters or *quartaroli* of wheat, or of another flour, and so on in all the subjects occurring in money, weights and measures, geometrically or not geometrically, as it has been said about the natural unit. This is why this sort of number can be called natural numbers, or denominate numbers.

But the Mathematician considers the above mentioned number as a multitude composed of Mathematical units, that is abstract from any sensible matter, according to reason, that is indivisible according to quantity, and such a species of number is conveniently called mathematical number, and this is also stated by Aristotle in the first of the *Metaphysics*, 38th text. It is true that the Commentator assigns three species of numbers, the first being the Mathematical number, stating that this cannot be increased or multiplied by multiplication of numbered things, because such a number, because it is considered abstract, is infinite. The second is a formal number, which diversifies itself by numeration of things which exist, and that the third is such that the previous is its form, and this third one is the number of sensible things, that is what we call natural number. But because my intention is not to clarify what is formal number and material number, because it is not the thing we have to deal with, I skip this to abbreviate. Also Albertus Magnus and Michael Scot and similarly Peter Lombard say that there are three sorts of numbers, and not more, that is *Numerus Numerans*, *Numerus Numeratus* and *Numerus Numerabilis*. The Number numbering, they say, is our soul, which numbers things by means of the mouth, the tongue and the hearth. The numbered number, they say, are the numbered things, such as animals, money, and other matters which are bought and sold by number, weight and measure and this sort of number is that which we call natural Number, the numerable number, by which we shall number. They say that it is the use and the act of numbering in various things, that is that discrete quantity which is required as a multitude and which starts with a unit as are these: one, two, three, four, five, six, seven, eight, nine, and so on to the infinite. This is what we call Mathematical number (being however abstracted from any sensible matter). From these derive four other generations, as says Isidorus, the first of which begins from the unit, and lasts until number ten, which is called number of units. The second is called number of tens, because it starts from ten and lasts until a hundred. The third is called number of hundreds because it starts from a hundred and lasts until a thousand. The fourth is called number of thousands because it starts from a thousand and proceed to the infinite. It is true that our modern practical (mathematicians) have added another fourth generation to them, which is called number of millions, which means a thousand thousands, that is a thousand times thousand. And this of millions together with that of thousands proceed later to the infinite, as before the act of numbering will become clear in the following book. *General Trattato*, Prima Parte, f. 2v)

I reproduced these passages at length in translation because they have not been published since the original edition of 1547. We retain from these passages that Tartaglia's interpretation of number is number as a number of things, and this in a sense is quite different from what Klein says, that the Greeks have an idea of number as number of things. Tartaglia indicates the steps from Aristotle's idea to his own. After Aristotle, number is a number of things in the sense that it is a multitude of abstracted units, which doesn't make sense without the act of counting units. Consistently, the very definition of unit implies indivisibility, as it implies that the units are homogeneous. In Tartaglia's vivid description of the merchant's world in the Venetian Republic, the idea is that one means "one of a kind," with its multiple and submultiple. By consequence, things are countable and comparable in any possible way. Merchants count and compare numbers, weights and measures of goods, fields. Tartaglia studies barter in detail because the mathematicians provide the reduction of anything to money reassuring that the transaction is fair. In the end, one sees things, in fact, the goods themselves, translated into measures. Numbers, degrees, inches are taken as things. In this context, number is the formal version of money: discrete, because subdivided in definite parts, like the submultiples of a currency, but also continuous because any unit of measure and any number can appear in calculation, and approximation is often necessary. Hence, we learn from this passage that Tartaglia considers the units of the Natural as divisible, in fact, infinitely divisible. Because units are understood as things, and things are infinitely divisible. In this perspective, also discrete quantities are (potentially) infinitely divisible. They are all measurable, that is, divisible and reducible to (some) units. In this sense, the distinction between discrete and continuous quantities becomes irrelevant for the Natural, because all quantities are continuous. Tartaglia can found his argument on his deep knowledge and experience in Euclid's *Elements*: his Italian edition and commentary makes use of previous interpretations. He added his own view in *Seconda Parte* of the *General Trattato*: in it, he introduces algebra in a Euclidean setting: after a new definition of number, of roots and of number as roots, he even develops in detail an algebraic interpretation of Book II of the *Elements*. In the passages above, Tartaglia argues for new notions in connection with practical arithmetic, which is his main concern in Verona's abacus school. But all mathematics comes to be interpreted from this perspective. In a similar context, Stevin will claim that number is continuous, as explained in the following chapter. Notice that in fact Klein deals with these questions, but only in connection with Stevin.

Tartaglia acknowledges that he bases his very innovative outlook on three authorities: Aristotle, "The Commentator" Ibn Rushd and Savonarola. They all support Tartaglia's distinction between the Natural and the Mathematician, but the interpretation of the unit as number and in this sense always divisible can be ascribed to Tartaglia, with the inspiration of a pseudo-Boethius which had been a reference for the algebraist Luca Pacioli (1445–1517), as we shall see below.

12.3 Ibn Rushd

Explicit mentions of Ibn Rushd's *Commentary on the Metaphysics*[10] and of Girolamo Savonarola's *Compendium totius philosophiae tam naturalis quam moralis. Opus de*

10 Aristotelis, *Metaphysica, latine, cum commentariis Averrois*. Padova, Lorenzo Canozio per Giovanni Filippo Aureliani, 1473.

divisione, ordine ac utilitate scientiarum[11] remind us explicitly that Tartaglia lived in the Republic of Venice. Both authors were not seen as harmless by the religious authorities, the first being Muslim and criticized and condemned by many Church Doctors, and the second burned for heresy in Piazza della Signoria. However, the *fortuna* of these authors in the printing press of the Republic of Venice is remarkable. Ibn Rushd's *Commentum magnum* to Aristotle's *Metaphysics* and to Aristotle's *De Caelo* in Guillelmus de Moerbeke's and Michael Scotus' (another author mentioned by Tartaglia) translations had appeared in Padua as early as 1473, only twenty years after Gutenberg's Bible. Not surprisingly, Averroism still had a strong impact on the University of Padua and will continue to have it for some time. Averroes was confirmed as an authority, The Commentator.

Aristotle's *De Caelo* begins by distinguishing three kinds of magnitudes: lines, surfaces and bodies. Ibn Rushd's *Commentary* specifies that the point of view of the natural is to consider things in material bodies, whereas the mathematician considers these magnitudes as abstracted from matter. As we have seen, Tartaglia echoes this choice. Ibn Rushd writes:

> The mathematician considers in magnitudes that which are the things abstracted from matter, whereas the natural considers in them the things which are in the ultimate matter.[12]

Concerning unit and number, the source is the Great Commentary to the *Metaphysics*, starting with quantity. He distinguished the various uses of the category of quantity. (Notice that in this English edition, square brackets indicate inserts.)

> 'Quantity' is predicated of all that is measurable by a part of it. Primarily [and] properly it is predicated of number, then [also] of the other genera mentioned in that book. There are essential as well as accidental quantities. Essential [quantities] are like number and the other species mentioned [in the Categories], accidental [quantities] are like black and white, since they are measurable inasmuch as they [occur] in a spatial extension. Essential [quantities] can occur in a thing primarily, like the measurability of number or spatial extension, and they can occur secondarily and by means of something else, like time which is reckoned among quantities solely due to [its connection with] motion, and motion [in turn] due to [its connection with] spatial extension. In a yet more extended [sense] heavy and light are included among quantities, since they are qualities and measurable only inasmuch as they [occur] in things with spatial extension. Almost the same applies also to other qualities which [occur] in things with spatial extension such as the large, the small, the narrow, the wide, and the deep. Although these are similar to qualities, they are nevertheless reckoned among the quantities because they are existents which occur primarily in things with spatial extension.
>
> (*Epitome* p. 32)[13]

11 Venice, Aurelio Pinzi, 1534; Venice, Giunta, 1542.

12 *De Caelo*, Padua, ed. cit, I, 1.

13 Rudiger Arnzen, *Averroes on Aristotle Metaphysics*. Berlin, 2010. From now on, *Epitome*.

This description of quantity is very inclusive and opens the way to all mathematical arts or mixed sciences developed particularly during the Middle Ages.

Further, Ibn Rushd distinguishes many possible meanings of the one. He writes in the *Epitome*:

> 'One' is predicated in one of the ways [we use] terms predicated with reference to one thing. The primary [way] to predicate ['one'] in this [sense] is the numerical 'one', the commonest [use] of which [applies] to the continuous, as in speaking of one line, one surface, or one body. What is even more appropriate among these [modes of predication] to be called 'one' is that which is perfect, i.e. that which does not accept any addition or subtraction, such as the circular line and the spherical body. The continuous can be continuous by imagination, like line and surface, or it can be continuous by something in it, as in the case of homeomeric bodies (in this [meaning] we call a concrete [mass of] water 'one'). We also predicate 'one' of that which is connected and contiguous (this is that which has one motion), and even more so of that which is connected by nature (these are things grown into one, such as one hand, one leg), and of these
>
> [especially] those which have only one motion. In another way, ['one'] is predicated of that which is connected by art, such as one chair, one cupboard. Furthermore, 'one' is predicated of individuals which are one by form, such as Zayd and Amra. Now, these are the commonest meanings of predication of the numerical.
>
> (p. 35)

We see here the authoritative origin of the notion of one as a continuous thing. In particular, this justifies Tartaglia's multiplicity of cases, which are the declension of variety of continuous things.

He clarifies how the number emerges as a collection of units:

> This then is how [the concept] 'one', which is the principle of number, emerges in the mind, for when the intellect abstracts from [the apprehension of] these individuals this meaning which cannot be split up into two or more individuals, this will be the 'one' which is the principle of number. When the intellect then uses [this concept] repeatedly, the [concept of] number emerges.
>
> (*Epitome*, p. 37)

And later, page 41 of the *Epitome*, he insists on the notion of one with respect to numerical quantities, in an ongoing dispute with Ibn Sina (980–1037):

> the term 'one' is predicated of the indivisible first in each genus by predication with reference to one thing, and most appropriately of that which is first in this [way] by being the cause of unity in substances and by being that which assesses and measures the one in numerical quantities. The numerical one is either indivisible by form and divisible by quantity, as [in the case of] one man, one horse, or it is indivisible by quantity and form. The latter [occurs] in two ways: if it has position, then it is a point; if it has no position, then it is the universal one, which is the principle of number and the essential notion of all that is countable. For all notions similar to this are only analogical [notions], such as measures or the weight unit [called] and the like.

This appears to be the passage in Ibn Rushd which motivates Tartaglia's sentence we saw earlier concerning the notion of point. In that sentence, Tartaglia mentioned his evolution from his version of Euclid's *Elements*[14] to the *General Trattato*.

Further in the book, we also find: "*It is, however, evident that number must be in matter and that it has unity only due to form, and multiplicity due to matter*" (p. 84), which is what Tartaglia expressed by saying that number is for things what soul is for animals.

In conclusion, Ibn Rushd emphasizes that "*The numerical one is that which is concrete in the mind*," and more generally that magnitude is in things, number in the mind. Furthermore, in his *Epitome*, he criticizes Ibn Sina for not understanding this typical view and its consequences.

> If there were no soul, there would be no numerical oneness and no number at all. Things are different in the [case of] the [individual] line, surface or, in general, continuous quantity. Therefore, number is more remote from matter [than the quantitative unit].

A point relevant to our reading of Tartaglia ascribing his view on unit and number to the Averroistic tradition is the fact that Ibn Rushd himself, in the *Epitome* on *Metaphysics*, comments on the corresponding passage by Aristotle (and mentioned by Tartaglia) about the three theoretical sciences. He distinguishes between the mathematician, the physicist and the metaphysician, in a way similar to the comparison between the mathematician and the "natural" (which seems to come rather from the *Commentary* on the *De Caelo*):

> Likewise, it is self-evident that mathematics abstracts this meaning from the individual substrate and considers it in its own right (just as it abstracts line, surface, and body). In this [respect] the consideration of the [one] by him who practices this science [of metaphysics] is different from that by the mathematician. For he who practices this science considers it in so far it is one quantity or one substance, while the mathematician considers it only in so far it is one quantity, abstracting [it] from any substrate. Similarly, the physicist considers line and surface in so far as they are the limits of the physical body, while the mathematician considers them only in so far they are line and surface. If this is the case, one and multiplicity pertain to the objects of consideration of both him who practices this science as well as the mathematician, while their considerations of these differ in respect (as is the case with different disciplines considering one and the same subject matter).
>
> (p. 114)

Notice that Ibn Rushd clearly follows Ibn Sina about the distinction between *prima* and *secunda intentio*. He writes, always in the *Epitome*, "*second intentions, that is intentions that exist only in the mind*" (Chapter I, p. 29). He develops this distinction only partially; then, he refers to his commentary of the *Posterior Analytics* which, however, is not mentioned by Tartaglia:

14 Niccolò Tartaglia, *Euclide Megarense philosopho*.

> Somebody might doubt whether the universal is of this kind, saying that universals must be accidents, if we suppose them to be mental and [raising the question] how they can make known the self-constituted substances of concrete things, when they are [mere] accidents, having said that that which makes known the quiddity of a substance is substance. However, this doubt is easily dispelled. For when the intellect abstracts these forms from matter and thinks their substances according to their true being, these forms, no matter whether they are substantial or accidental, adopt in that state in the mind the meaning of universals. [This does] not [mean] that the universal is itself the form of these essences. For that reason, universals are second intentions, while the things of which they are accidents are first intentions (the difference between first and second intentions has been stated in detail in the discipline of logic). All this is self-evident for those who practice this discipline.

It would be a whole project to study the reception of The Commentator on *Posterior Analytics* in Italy and especially in the Venetian printing, along with its context from the end of the fifteenth century to Tartaglia. For now, let us just recall this intensive reading of Ibn Rushd's *Commentaries* and of all their variants, in manuscript from commonplace books, but also printed, from the few ones published around 1480 by Manutius to the great editions by Giunta in 1543 and 1562.

In fact, there are many reasons explaining Ibn Rushd's influence on Tartaglia. First of all, Tartaglia was a citizen of the Venetian Republic, a teacher at the abacus school in Verona. Within the Republic, and especially at the University of Padua, Averroes' commentaries of Aristotle had been present in manuscript at least since 1225. More recently, they had been printed together with Aristotle's works, as we have seen, in the first decades since the introduction of printing. In fact, in the sixteenth century, they were the common source for the study of Aristotle. In fact, Padua's famous Aristotelianism was mostly Averroistic. This continued until 1630.

While Tartaglia did not belong to the milieu of the university, his frequent references to *the Commentator* show that he relied on Averroes' authority, so we can formulate the hypothesis that this interest in Averroes was also reinforced by his familiarity with the abacus school tradition. In fact, another author mentioned by Tartaglia is relevant in this connection: the Medieval polymath Michael Scot (ca1175–ca1232), who belonged to the second generation of Toledan translators of scientific works. As we have mentioned, he was responsible for the translation of Averroes' commentaries,[15] in particular of the *De Caelo*, which is precisely the source of the distinction of mathematical and natural knowledge expressed by Tartaglia. But Michael Scot was also the dedicatee of the second version of the *Liber Abaci* by Fibonacci. Tartaglia himself mentions Michael Scot as an authority together with Albertus Magnus in the section on number. Therefore, when Tartaglia used this passage, he was aware of inscribing his thought on mathematics in the tradition of Averroes and Michael Scot, a syncretic tradition in philosophical, theological and mathematical terms. This syncretic tradition included, for instance, a new vision of the mathematical sciences, with a

15 In fact, according to Dag N. Hasse, he was responsible for the translation of all of the most important commentaries: on *Metaphysics*, *Physics*, *Posterior Analytics*, *De Anima*. Hasse, Dag Nikolaus, "Influence of Arabic and Islamic Philosophy on the Latin West," *The Stanford Encyclopedia of Philosophy (Fall 2008 Edition).*

redistribution of the roles between theoretical and practical arts which can be ascribed to Al Farabi, and this vision included algebra. More generally, it included a perspective on human knowledge and an impulse to develop a language and a logic common to this new encyclopedia.

12.4 Gerolamo Savonarola

Tartaglia mentions Savonarola, known mostly as a preacher and a political figure, as an intellectual authority. In fact, his compendium of Aristotelian philosophy and in particular his logic were important in the sixteenth century, as Richard Popkin has shown in his classical work on the history of skepticism. He reports that Savonarola taught logic in the monastery of San Marco, and that many Florentine intellectuals gathered in the library of the monastery, where five manuscripts of Sextus Empiricus were present, including the one belonging to Pico. This is an example of how we should consider Savonarola as belonging to humanistic circles interested in Ibn Rushd and in mathematics.

Tartaglia is referring to Savonarola's work concerning the division of the sciences.[16] Savonarola distinguishes between the mathematical sciences such as arithmetic and geometry and the other mathematical sciences depending on them: the ones Aristotle had talked about in the *Posterior Analytics*. Savonarola starts from the objects, which, being proper to mathematics, are abstracted:

> Hence, either we consider things as abstract and without any position, and in this way we have the unit and numbers, studied by Arithmetic, or they are thought of insofar as they have a position, and in this way we have magnitudes, about which Geometry is built. But among mathematical sciences there are some intermediate ones, such as Astrologia, which deals with the motions of the stars, and perspective, which deals with visual line and magnitude. And Music, which takes into consideration sound numbers and their proportions. Therefore, these sciences are mathematical insofar as they make use of mathematical principles, and statements and proofs. As to the object, they are natural, insofar as they deal with natural things. And because object is more about the essence of a science than its way of practicing it, in my opinion they should be rather called natural than mathematical.
>
> (p. 4)

In this "division of sciences," Savonarola takes a precise position with respect to the issues of mixed sciences. These are the ones, as he says, which use mathematical principles, statements and proofs, but their object is natural, and is not abstracted from things. This responds to a debate current at that time, developed by Tartaglia himself at the beginning of his translation of Euclid's *Elements*. Aristotle's view had been hierarchical: after giving priority to arithmetic and geometry, which should be distinct, that is independent from each other, at least a subordinate science stems from each of them: e.g. astronomy from geometry and harmonics from arithmetic.

16 *Opus perutile de divisione, ordine ac utilitate omnium scientiarum Reverendi P. Fratris Hieronymi de Ferraria, nunc primum in lucem editum*. Aurelio Pinzi, Venice, 1534.

This Aristotelian doctrine is mostly explained in *Posterior Analytics* I, 7 and in *Metaphysics* 13, 3. However, this classification had been first corrected by Proclus, then by Boethius and during the Middle Ages went through a series of variations and developments to include a large number of disciplines which had become very important, especially in Arabic. The tensions were due to logical problems, such as whether these subordinate sciences had the same certainty as mathematics, which depends on its logical aspects, but also on the fact that its objects are intermediate, that is immovable and not independent but only separated by abstraction. Therefore, they do share the logical aspects, but not the ontological status of their object. Savonarola states clearly that these intermediate sciences are closer to physics than to mathematics, because while their method is mathematical, their objects are in natural things and what counts is the object. This stresses on the crucial role of the object and not on the practice to determine the essence of a science; for the central role of mathematics as well as for the commitment to give a relevant status to the mixed sciences, Savonarola is clearly inspired by the Medieval tradition which can be traced back to Al Farabi.[17] Al Farabi's *Enumeration of the Sciences (Iḥṣā' al-'Ulūm)* was accessible in Latin through the version written around 1150 by Gundissalinus, that is, Domingo Gundisalvi (1115–1190), the Toledan philosopher, mathematician and translator of the twelfth century, and later, around 1180, by Gerard of Cremona (1114–1187). Gundisalvi has actually two texts strictly connected with Al Farabi's *Enumeration*: *De Scientiis* and *De Divisione philosophiae*. Gundisalvi's versions were much less faithful to the original, but had a much greater influence in the Latin West.

Savonarola taught logic: the book we just cited was published in Venice in 1534 together with the *Compendium totius philosophiae tam naturalis quam moralis Reverendi P. Fratris Hieronymi Savonarolae de Ferraria*, and will be reprinted by Giunta in 1542. This is a summary of philosophy. But we can find in it a series of elements which could make the text particularly interesting for Tartaglia. Savonarola begins with logic. He writes that

> *Logic is rational and opens a way towards the principles of all methods*. This is a paraphrase of the definition by Petrus Hispanus in the *Summulae logicales*, the often reprinted textbook of Aristotelian logic. In spite of this role of logic, says Savonarola, many religious people are afraid of it. This is why his program is to deal with the whole of dialectic *more mathematico*: "I decided to give a synthesis more mathematico of all dialectic according to the excellent authorities shortly, distinctly and easily, so that all those who want to learn it can have an easy plain way to it."[18]

In other words, we have here a true book on dialectical method of a type which will start becoming popular in Europe. The Florentine humanist gives here the definition of "vox," that is word as pronounced.

17 See Lisa Jardine, *Francis Bacon. Discovery and the Art of Discourse*. London, 1974, p. 98.

18 "statui breviter, distinctae et faciliter totam dialecticam ex dictis excellentissimorum virorum **more mathematico** ita in unum colligere, ut omnes qui eam perdiscere cupiunt, praetermissis obscuritatibus facilem ad eam viam habeant et planam." *Compendium Logicae*, p. 2.

> People create words to express the mind's concepts to each other. People create letters to preserve the words that would normally disappear.

Later:

> A word of first intention is that which means a concept immediately captured from the thing, and is similar to it and founded immediately on it.

This is important for us because we find, in an author mentioned directly by Tartaglia, a definition of first and second intentions, stressed by Jacob Klein in connection with the meaning of the *species* in Viète and Descartes.

Savonarola explains the definition at length:

> For instance this word *homo*, which has been invented by Latin people to express the concept of human nature and which is taken directly from human nature and is similar and as an image of it, is founded immediately on it, because human nature is its immediate object.

Instead:

> The term of second intention is the one which means concept of the mind, immediately taken not from the thing understood, but from a way of understanding the thing, and is immediately founded on the first concept of the same thing.

More precisely,

> For after that the mind conceives human nature by the first intention, the mind starts to compare it to individual human beings and as it sees that it is appropriate to all individual human beings. Then it forms another concept of human nature, because first it considered human nature in itself, whereas now it considers it according to what it is in many or in all human beings. This concept is called **second intention**, because it is taken not immediately from the thing, which is human nature, but rather from that way of understanding by which the mind understood that it was appropriate to all human beings.

Savonarola explains this definition:

> Hence the minds calls it a universal or a *species*. By consequence, these words: genus, species, difference, proper, accidens, individual, universal, predicable are words of second intention.

That is, Savonarola uses the Avicennian and Averroistic theory of *prima et secunda intentio* in the context of a humanistic dialectic. He does not discuss it as a commentary on *Metaphysics*, or in the metaphysical section, but takes it at the most fundamental level of dialectic, that is at the level of the *vox*, and of the relation between words and things. Here, we see a trace of the humanistic side of Savonarola, the connection with the new intellectuals that took form in the meetings with Poliziano and Giovanni Pico della Mirandola. Published about 40 years later in Venice, after Savonarola's execution at the stake, his theory was less controversial (it concerned classical sections of dialectic, not theological or political points) but carried with it the substantially innovative rhetorical turn in logic typical of the humanist movement.

Very recent research[19] by Hanna Gentili indicates the presence of the Hebrew intellectual Alemanno in this same "Dominican" circle. Alemanno is relevant here because of the interpretation of the *Posterior Analytics* and other Aristotelian canonical books based on Ibn Rushd, as well as for the curriculum of studies he proposes, again with an extended group of mixed sciences.[20]

I have shown elsewhere[21] that vernacular humanism was connected with the rhetorical turn in logic and dialectic on the one hand and with the development of algebra in the sixteenth century on the other. Vernacular humanism promoted the publication of literary and scientific books in the vernacular and algebra had been mostly transmitted in the vernacular abacus schools. Hence, after Arabic, Hebrew and Latin, its language had been Italian, German, Spanish, Dutch and English. Vernacular humanism appears in these explicit references in Tartaglia's text, and we shall also see them in the main Italian authority in algebra until Cardano and Tartaglia, that is, Luca Pacioli. Tartaglia points to Savonarola, an author belonging to Florentine humanism, and to Ibn Rushd as the authorities justifying the novelties introduced by sixteenth-century algebra: the themes are unit, number, theory and practice, "pure" mathematics versus sciences using mathematics, and dialectic, as well as first and second intentions, which was so stressed by Klein in his essays. The discussion about the division of science will appear in most algebraic treatises: the point was critical, given that algebra had no classical place in the classical Latin tree of sciences. Algebra had a place only in the one proposed by Al Farabi: in his *Enumeration of the Sciences*, he had also given priority to dialectic and extended the mathematical sciences to the disciplines of algebra, optics, statics and *de ingeniis*.

12.5 Luca Pacioli's project and philosophical background

In the previous section, we have seen an example of the Dominican side of Florentine humanism: Girolamo Savonarola and his monastery. We have seen how the monastery's library could receive major intellectuals such as Giovanni Pico. We turn now to another intellectual circle, including Marsilio Ficino, connected to a religious order, the Franciscans and its most important algebraist: Luca Pacioli.

Luca Pacioli's *Summa de arithmetica, geometria, proportioni et proportionalità* (Venice, 1494) is not only a rich source for Medieval commercial mathematics, but also an influential book that will be used, cited and criticized for a long time because of its way of dealing with algebra. We shall see some of its philosophical points and

19 Hanna Gentili has presented this discovery at my seminar at the EHESS in 2019.

20 Moshe Idel, "The Anthropology of Yohanan Alemanno. Sources and Influences." *Topoi* 7 (1988): 201–210.

21 Besides *Mathematics and Rhetoric*, cit., see *The Art of Thinking Mathematically*, special issue of *Early Science and Medicine* 11(4), 2006: 369–477, "From Valla to Viète: the Rhetorical reform of Logic and its Use in Early Modern Algebra", pp. 390–423. For the dialectical reform by Lorenzo Valla, see the classical Lisa Jardine, "Lorenzo Valla: Academic Skepticism and the New Humanist Dialectic," in Burnyeat (ed.), *The Skeptical Tradition*, 1983, pp. 253–286 and the more recent Lodi Nauta, *In Defence of Common Sense: Lorenzo Valla's Humanist Critique of Renaissance Philosophy*. Cambridge MA, 2009.

their consequences on some of Pacioli's main readers on the sixteenth century. Here, his book will help us look at the main source of algebra as well as of its philosophical content.

In his dedicatory letter of the *Summa* to Guidobaldo duke of Urbino, Pacioli elaborates on the importance of number in the Creation and of the mathematical sciences in particular. Here, we find his definition of quantity:

> Quantity, magnanimous Duke, is so noble and excellent that many philosophers considered it equal to substance and coeternal to it. This is why they established that nothing can exist without it in the nature of things. This is why I plan to deal with this topic, with the help of The One who governs our senses. This is not because other authors did not write much about it, in theory and in practice. The reason is that their texts are nowadays very obscures and very poorly understood or learned by the majority of people, and also that they are badly applied to practical use. In fact, their operations vary greatly and only with great labors are they put to work in number and measure. This is why when I talk about quantity I do not mean anything but that which is useful to practice and to operate. Yet, I am going to add some theoretical passages and the cause of calculating, of number as well as of geometry. But in order to be able to learn the following we shall divide quantity according to our purpose, and by dividing it we shall assign to each term its true and proper definition and description.

In the first sentence, Pacioli is referring to the Book of Wisdom, XI, 20, a reference he made explicit in another of his works, *De Divina proportione*: *God has regulated everything by number, weight and measure*. He claims that many people wrote about quantity, but they failed when it came to make this knowledge useful in action, so that to apply the content of mathematical texts becomes too hard. Hence, he will concentrate his efforts on this and add only a few theoretical points and clear definitions.

> First we shall distinguish the different kinds of quantity and (…) we shall give it its proper definition and description. Then will follow what Aristotle says in the second book of *Posterior Analytics*, because then *Tunc enim maxime scitur aliquid cum habetur suum quid est* etc. (we shall really know something when we know the fact, the reason why, if it is, what it is)[22]
>
> (*Summa*, f. 1)

Here, we see that Pacioli makes explicit the scientific character of his *Summa*, insofar as it will follow the rules of Aristotle's logic. Here, we see immediately the pertinence of *Posterior Analytics* with early modern algebra, to stress its scientific standards.

Pacioli then moves to the crucial distinction:

> I say that quantity is immediately twofold, that is continuous or discrete. Continuous is the one the parts of which are connected by a certain common term,

22 A reference to *Posterior Analytics* II, 1: "We seek four things: the fact, the reason why, if it is, what it is." (Here in the translation by J. Barnes, Oxford, 1975, p. 53). Four editions of Aristotle's *Posterior Analytics* had been published in Venice before 1494, translated by Giacomo da Venezia and with commentary by Ibn Rushd.

> such as **wood, iron, rock** etc. The discrete is the one the parts of which are not connected by a common term, such as 1, 2, 3 etc. We shall deal first with the purpose of the discrete in the arithmetic and then of the continuous in the geometry.

Then comes the definition of number:

> Number according to all philosophers is a multitude composed of units. And this unit is not number, but rather the principle of each number and is that by which each thing is said to be one, as Boethius wrote in his *Musica*: the unit is any number potentially and later in his *Arithmetica* Boethius calls it queen and foundation of any number. And by magnifying it even more in natural things he says in the book entitled *De Unitate et Uno*: ***Omne quod est ideo est quia unum numero est.*** (Everything which is, is because it is one in number)

A first remark is that the last sentence comes in fact from Boethius' *Commentary to Porphyry's Isagoge*. Notice that Pacioli attributes this passage to Boethius' *De Unitate et Uno*. In fact, recent research has proven that the author of *De Unitate et Uno* is not Boethius but Domingo Gundisalvi. What we find in Gundisalvi's *De Unitate et Uno* is not quite the same sentence, but a very similar one: ***Quidquid est, ideo est, quia unum est.*** (Anything that is, is so because it is one).

Pacioli will repeat Boethius' sentence, again with the inappropriate reference to the *De Unitate et Uno*, at the beginning of the algebra section of the *Summa*. Apparently, it is at the core of his view on the topic. In fact, it seems that the whole of Gundisalvi's treatise is a precious source for Pacioli. In his *De Unitate et Uno*, Gundisalvi proposes a whole theological ontology based on the One, which also establishes a new role of unit among discrete and continuous quantity. In fact, this book put under Boethius' authority carries a very interesting blend of Greek (Proclus), Arabic (Al Farabi, Ibn Sina, Ibn Rushd), Jewish (Ibn Gabirol 1021–1070) and Latin philosophy of mathematics.[23] After the sentence mentioned above, Gundisalvi also writes that:

> Anything which is, is either one or many. Yet, plurality is nothing but an aggregate of units. Disaggregated units make a multitude, whereas continuous units in matter make magnitudes. This is why there is nothing different between the units of discrete and the units of continuous quantities existing in matter, if not that those are disaggregated, while these are continuous. By consequence, continuum is only from disaggregated, because understanding of continuity in continuous is nothing but continuation of disaggregated. Because of this whatever part of matter you will express is by necessity one or many. But any plurality, as we said, is made by units. Hence, it is clear that there is **one and only one root of discrete and continuous quantity**, so that the composed things are from one thing and are resolved in one. The more the parts of the body will be connected and linked, the more the body itself will be thick and of **more quantity, like rock,** so to the contrary, the more the parts of the body will be dissolved and rare, the more the

23 Nicola Polloni, *Ontologie divergenti: uno studio sul sincretismo metafisico di Gundisalvi*, PhD thesis, Pavia 2015. For the mathematical implications, especially concerning algebra, see Gad Freudenthal, "Abraham Ibn Daud, Avendauth, Dominicus Gundissalinus and Practical Mathematics in Mid-Twelfth-century Toledo." *Aleph* 16(1) (2016): 61–106.

body itself will be subtle and light and of less quantity, like air. It is true, therefore, that continuous quantity does not come in substance if not in conjunction with the strictness of units within it. In conclusion, **unit is that by which anything is one and is what it is**.

[*Unitas igitur est, qua unaquaeque res est una et est id quod est*.]

This is the conclusion of Gundisalvi's text. It is a demonstration of the common root of discrete and continuous quantity, and this by means of a notion of unit which links the discrete and continuous. Here, Gundisalvi seems to refer implicitly to *Metaphysics* XI, 12, in which Aristotle distinguishes between the being consecutive of units in a number and the being contiguous of points on a line (we have seen this discussed by Tartaglia). We can also see in it a reference to the numbering feature of unit, that is, the fact that adding a unit makes infinity. This is what is also suggested by the rest of Pacioli's letter:

> Number is divided into infinite members, this is why Aristotle says: «*si quid infinitum est, numerus est*» [if something is infinite, it is number]. And because of the third petition of book VII of Euclid, its series can proceed to infinity. Given any number, we can get a larger number by adding a unit. But we shall take the parts better known and more appropriate for us. This is the reason why some number is prime with respect to others and is the one which is numbered only by the unit, and not by any other number which would divide it without remainder. Composed number is that which is measured or numbered by another number.

Although the reference is to Aristotle,[24] the sense of the first statement is part of the interpretation due to the Neo-Platonic tradition. Plotinus (204–270), for instance, discussed the inexhaustible character of number, which is also the point made by Pacioli.

In the whole passage, Pacioli refers very closely to the beginning of Book VII of the *Elements*, whereas most algebraists and abacus writers in general did not mention Euclid directly. Pacioli was in fact a Euclid scholar. Born in Sansepolcro, in 1464, he had left to join the Rialto school in Venice, a school instituted in 1408 which combined abacus teaching for merchants and some liberal arts studies, propaedeutic to the university. Starting in 1569, Venice became quickly the main center of the printing press. Euclid's *Elements* became the scientific bestseller of the time. In Venice, the Augsburg's printer Erhard Ratdoldt published the first edition in 1482, based on Campanus' version, reprinted the same year.[25] In 1491, Leonard of Basel printed Euclid's *Elements* in Vicenza, since 1404 under the political rule of the Serenissima. Besides these two editions used for writing the *Summa de arithmetica*, Pacioli certainly also read Zamberti's version of the *Elements*, the first edition correcting explicitly Campanus' version, printed in Venice in 1505 by Giovanni Tacuino, because Pacioli himself lived in Venice a few times in his life. In 1509, Pacioli published in Venice his own version of Euclid's *Elements*, as well as his *De Divina Proportione*

24 Though in *Physics*, I, 2 (185b3), we find that the infinite is in the category of quantity.

25 *Preclarissimus liber elementorum Euclidis perspicacissimi : in artem Geometrie incipit quam foelicissime*. Erhard Ratdolt, Venice 1482.

decorated with Leonardo da Vinci's polyhedra with the printer Paganino Paganini, who had already printed his *Summa de Arithmetica*.

To come back to Book VII of the *Elements*, this is the book in which Euclid provides definitions and proves theorems about numbers and numerical proportionality. The first definition cited by Pacioli is ***Naturalis series numerorum dicitur in quam secundum unitatis additionem fit ipsorum computatio.*** [We call the natural series of numbers that in which the computation is made by addition of the unit.] Pacioli here depends on Campanus' version of Euclid,[26] as will do all editions until the *editio princeps* of the Greek printed in Basel in 1533. Campanus translates Euclid from Arabic but includes mathematical aspects of various twelfth-century texts. He introduces some new *petitiones* or hypotheses on numbers, one of which is precisely that "the series of numbers can proceed to the infinite." This whole group of definitions and hypotheses and even common notions establishes in the West the interpretation explicitly connecting Books II and VI, that is, the books concerning applications of areas,[27] with the arithmetical Books VII–IX where properties and proportions with numbers were represented by segments, plane and solid numbers. As we have seen at the beginning of this chapter in connection with geometric algebra, in Book VII, Campanus adds the notable products, the arithmetical counterpart of the first ten theorems of Book II. So, the association between arithmetic and geometry (following the same laws) develops into an association between arithmetical and geometrical proportions. Notice that Campanus knew Fibonacci's work. This interpretation of Euclid is fully exploited by Pacioli as the basis for the foundation of what he describes as the main rule of algebra. This, in turn, will change arithmetic and geometry.

From the philosophical point of view, it is relevant that Pacioli mentions Gundisalvi, because from the original Aristotelian idea of a single being as one, we come to the idea of being one in number (*unus numero*) that is implicitly considering one as a number. This means in fact to take the point of view that Ibn Rushd attributed to the Natural Philosopher. In ontology, Gundisalvi depends mostly on Ibn Gabirol's theory of generalized hylemorphism, a theory of matter[28] which confirms the connection we have seen in Gundisalvi between continuous and discrete quantities through the infinite sum of units: all this seems to be implied here. This is a foundation of algebra on arithmetic and geometry, and wisely he mentions the *De Unitate et Uno*, because that is precisely the text which provides a proof that there is a common root of discrete and continuous quantity. This root is the unit, meant as number and the operation of successor. We shall see that this idea is shared and developed by a few sixteenth-century mathematical authors. After a few sections devoted to arithmetic, Pacioli comes to the algebraic part. At *Distinctio Octava, Tractatus quintus*, f. 144, Pacioli writes:

26 As it is his own version of Euclid's *Elements*: *Euclidis megarensis philosophi acutissimi*. Venice, 1509.

27 On the Pythagorean origin of this part of Euclid's *Elements*, see Oskar Becker, *Das mathematische Denken der Antike*, 1957. On this interpretation of Euclid by Campanus, on application of areas and on the relevance for algebra, see Unguru and Rowe and Leo Corry.

28 See Polloni, "Gundissalinus Application of al Farabi's Metaphysical Programme." *Mediterranea. International Journal for the Transfer of Knowledge* 1 (2016): 69–106. More recent references appear on his website, *Potestas essendi*.

> We arrived, with God's help, to the very desired point: the mother of all cases, called rule of the thing by the people, or the great art, or the **speculative practice** called Algebra and Almucabala in the Arabic language or Chaldean, according to some. In our language this means restauration and opposition, or contraposition and consolidation. In this way an infinite number of questions are solved, and those which are not yet solved are proved. And in order to learn well this treatise with the two following ones, all which has been said until now about numbers and their force (especially the extraction of all sorts of root), has been done for this. It is for this reason that the reader knows that he does not read in vain, because it is this part which teaches us everything, both in geometry and in arithmetic. Without its help infinite questions could not be solved.

Pacioli states here that algebra is the main goal of the arithmetic explained so far in the book. Note that he calls it *practica speculativa*, that is, theoretical practice, a practice that is in fact theoretical in character.[29] He distinguishes himself from many previous authors by mentioning the possible Chaldean origin of algebra, besides the Arabic one. He also tells us that algebra gives the means to solve or to prove the unsolvability of an infinity of problems, in arithmetic as well as in geometry.

He writes:

> Given that we want to talk about this part in order to reason on all sorts of things, and solve each case by a specific answer, I need to clarify these three quantities, or terms, which we need to put in the three chapters of the rule, that is *numero, cosa e censo.*

Pacioli continues:

> quantity carries always with it its determinate number because it is with each thing in nature. We see at the beginning of Boethius' work *De Unitate et Uno*: *omne quod est ideo est, quod unus numero est.*
>
> [all that is, is because it is one in number]

We recognize here the quotation from Boethius attributed to the *De Unitate et Uno* by Gundisalvi. Here, this is justified not only because the *De Unitate et Uno* contains a similar statement, but also because the point here is to insist that number is naturally implanted in things, born with each thing. In fact, the reference here is the beginning of Aristotle's *Physics*, where he says that human beings start to know the whole and then understand the parts.

Concerning the expression *Unus numero*, that is one in number, it is particularly pertinent here because algebra was called the art of the Thing, that is, the art that deals with the quantity of any Thing, given the relation of proportion that Thing has to give quantities. Gundisalvi's sentence means that the unit is in some cases a number.

29 For a different interpretation, see *Luca Pacioli e la matematizzazione del sapere nel RInascimento*. Bari, 2003.

Gundisalvi contributed to algebra in several ways. First, he is the author of a translation and commentary of Al Farabi's *Enumeration of the Sciences* as well as of his own *De divisione philosophiae*, inspired by Al Farabi. Both include algebra: problems treated algebraically are mentioned in connection with what is described as practical arithmetic or more precisely Mahamelet.[30] Furthermore, the word algebra is used in the chapter of *De Divisione* called *De ingeniis*, which we could translate as *Inventions* concerning constructions, as in our engineering. But Gundisalvi writes that this is the field of knowledge dealing with line, surface and body in matter, not only in the mind. This is reminiscent of the beginning of *De Caelo* we mentioned in connection with Tartaglia. He explicitly mentions the name of algebra as the artful way of solving problems in which we look for numbers. He concludes by saying that the numbers we are looking for can be rational, but often they are rational compared to (that is in proportion with) irrational numbers. Following Al Farabi, Gundisalvi attributed a special importance to *resolutio* in mathematics, that is, analysis as opposed to synthesis in proofs. Also, according to recent scholarship,[31] Gundisalvi played an important role in the network of scholars involved in the writing of the famous work *Mahameleth* which includes some problems solved by algebra. This book in fact collected already around 1145[32] the Medieval learning that will be included in Jordanus Nemorarius and Campanus' version of Euclid's *Elements*: especially al Kwarizmi, Abu Kamil, but also an arithmetical reading of the first ten propositions of Book II, together with some Nicomachus' arithmetic. All this material will are fundamental for algebra, and their combination became abacus mathematics, that is, commercial arithmetic, algebra and practical geometry later collected and explained in Pacioli's *Summa*. We also see that here Pacioli talks about any sort of quantity, and the problems he refers to are both in arithmetic and in geometry. Cosa, Censo and Cubo can be discrete or continuous quantities.

Together with the reference to Chaldaic knowledge, we see here a hint to the late fifteenth-century Neo-Platonic context of revival of Pythagorical beliefs transmitted by Arabic and Hebrew texts. As we mentioned earlier, these were translated and promoted in Florence by different groups of intellectuals, such as Marsilio Ficino to Pico della Mirandola and other humanists who were interested in Cabala and in Christian Cabala or simply interested in Hebrew as a primigenial language as well as to Chaldaic language.

In what follows, we find a reduction of everything to a number as a multitude of units, which has been probably the origin of what we read in Tartaglia:

> By saying this (Boethius) deduces what we have said, that is that any number is different from the other. And that the multitude of their units which is in the

30 For this work and for references to the secondary literature on it, see G. Freudenthal. See also Charles Burnett "The Coherence of the Arabic-Latin Translation Program in Toledo in the Twelfth Century." *Science in Context* 14, (2001): 249–288.

31 Charles Burnett, "John of Seville and John of Spain: a *mise au point*." *Bulletin de philosophie médiévale* 44, 2002: 59–78. Reprinted with corrections in *Arabic into Latin in the Middle Ages: The Translators and their Intellectual and Social Context*, Variorum Collected Studies Series, Farnham 2009, Article VI.

32 The reference here is Leo Corry, cited in footnote 6.

ducati is number as well as the multitude of units which is in the *soldi*. And the two numbers are different only by the multitude of their units because one has more and the other has less. And by consequence (in what is considered the algebraic practice) we can say that the number is the whole multitude of units of no matter what quantity you like arriving in the hands of the agent, be they *ducati*, *fiorini*, *bulls* or *horses*, and talking about all we can name by number. So we want it to be understood in all cases.

But this is also the introduction of number as part of the terms used in algebra. In other words, here Pacioli introduces the definition of the terms of equations: number, but also the powers of the unknown, that is, Cosa and Censo. The reciprocal relations of these powers are already compared to the reciprocal connection of the various divisions of money (e.g. pound and penny), as in the Arabic classical treatises, dinar and dirham.

What is meant by Cosa in algebra. We should not think that Cosa is without number; on the contrary, it has always with it its units and its own ratio. And similarly, Censo has its determinate and certain number.

By number we mean any multitude of units of no matter what quantity.

(f. 144 v)

Pacioli continues:

Any number can be considered in itself, as a simple number, or in relations to the multiplication by itself, and this in two ways, one, as the multiplied thing, and the other as the result of the multiplication. The *Cosa* multiplied by itself is said Root and is called *Cosa* by the practitioners. The product, that is the result, its number is called *Quadrato* and *Censo* by the practitioners. So these three forms are *Number*, *Root* and *Quadrato* or *Number*, *Cosa* and *Censo*.

Then, he abbreviates with Ce, Co and N. In our terms, this means:

Ce = Censo = x^2
Co = Cosa = Root (Rx) = x [in the Latin sometimes it is called « *Latus* », that is side
Number = known term

Pacioli explains:

<u>Concerning the equations between *numero, cosa* and *censo* just defined</u>. After having seen these terms for the arithmeticians, we have to see how many times we can equate one to another or one to two others: the first three are called simple equations, the last three are called composed equations:
Censo is equal to *cosa*
Censo is equal to *numero*
Cosa is equal to *numero*
Censo and *cosa* are equal to *numero*
Cosa and *numero* is equal to *censo*
Censo and *numero* is equal to *cosa*.

Pacioli presents these six kinds of equations of second degree (distinct insofar as there was no notation for canonical equations and coefficients were positive) or "chapters" by means of what we would call numerical examples of three equations, two simple and one composed.

We see that Pacioli has started using a series of rhetorical and graphical strategies to explain that the quantities appearing in equations are both arithmetical and geometrical. He gave a foundation to his argument by using three cases. These three cases correspond to equations which we represent in the following way:

Case 1: $5x = x^2$
Case 2: $4x = x^2$
Case 3: $3x = x^2$

These three equations have an obvious solution for us. In fact, they were also the basic cases for Pacioli. But they play at the same time very important role in the foundation of the discipline.

Here, Pacioli comes to the **first rule**:

> **First, there is no other way to put the equation besides these six equalities, called by the people the six chapters of the *Cosa*.**
>
> All the questions proposed are reducible to one of these or to some other in proportion to one of these. The first three are called simple, the last three composed.

5Co is equal to 1Ce → $5x = x^2$ (5 Cose is equal to 1 Censo)
1Ce is equal to 4Co → $x^2 = 4x$ (1 Censo is equal to 4 of its Roots)
1Ce plus 10Co is equal to 39 → $x^2 + 10x = 39$ (1 Censo plus 10 Cose is equal to 39)
Pacioli develops in particular the cases of simple equations of the type "**Cosa equal to Censo.**"
<u>5Co is equal to 1Ce</u>
Here Pacioli states the second rule which he will invite you to learn by heart:

> **Secondly as many Cose will be equal to one Censo as the units that will be contained in each of its Cose.** In other words as many Roots will be put equal to the Square as the units each Root will contain. For instance, five Cose are equal to one Censo, that is, the Cosa contains 5 units because 5 Cose are equal to one Censo. By consequence, 5 Censi will hold 5 × 5, which is 25, that is 5 of its Roots. This is because, as we have said, the Cosa has a root in the Censo [is the square root of the Censo]. So, in proportion, I shall say that of any other quantity which will be root of another. **And this you will learn by heart with the other**, and so on, for this, as many roots will be contained in the surface of a square as the units that will be in its roots.

We need to focus on this first rule. What appears here are continued proportions, the proportion we find at the very beginning of algebra as the square of the unknown, in our notation:

$$1 : x = x^2$$

which in Pacioli's terms became **there are as many unities in a cosa as there are values of the cosa in a censo.** Pacioli, maybe for the first time in the Latin West, provides an explicit Euclidean foundation of the rules of algebra, by justifying in both arithmetical and geometrical terms the progression of the powers of the unknown. As Pacioli puts it at the beginning of the algebra section, we have to commit to memory the principle that **there are as many unities in a cosa as there are values of the cosa in a censo.** This explicit association of values to a computation of segments will become a central principle in sixteenth-century algebraic books and in the mathematical foundation of proofs.

Pacioli is known as the mathematician who combined the practical mathematics of abacus schools with university mathematics, most notably with a generalized notion of proportionality. Pacioli's *Summa* shows that in general an equation can be seen as a proportion, and proportions are translated into equations. In his discussion of the quadratic equation, Pacioli maintains that proportionality is neither arithmetical nor geometrical, but instead situated deeply in the human understanding of common notions. For Pacioli, the first laws of proportion coincide with the first three common notions the he presents in Book I of the *Elements*. In fact, the first justification of the solution of quadratic equations in Pacioli's *Summa* is the explicit reference to the *communa conceptione* or common notion.[33] Already at the beginning of the *Summa*, Pacioli writes that the basic operations which apply to equations are presented in the title of the first Arabic algebra book, that is restoring (*algebra*) *and* opposing (*almucabala*). According to Pacioli, they depend on common notions which apply to both arithmetic and geometry. Restoring uses the second common notion, it is adding on both sides the negative term of an equation. Opposing is "simplifying" or subtracting a term on both sides of an equation and uses the third common notion. The first is the basic common notion and defines equality, that is, how to deal with two members of an equation.

Pacioli develops in this way the idea of algebra as a theory of "quantity," a concept that will be further elaborated much later by Viète and Descartes as general quantity. The classical reference was the computation of segments and "areas" resulting from their products that was attributed to the Pythagorean tradition and constituted a sort of primitive theory, existing before the differentiation of arithmetic and geometry and applicable to both. In Western algebra, among Pacioli's sources, we find explicit references to Book II as connected to algebra already starting with Fibonacci's *Liber Abaci*, drawing from Arabic authors, and Pacioli will develop this aspect with more references to actual theorems. Mathematically, one should think of a computation of segments independent of Book I and based exclusively on common notions and Book II, and therefore conceivable as "primitive" among mathematical sciences: the theorems are the remarkable products we know from algebra, but studied on segments. Most sixteenth-century authors will associate algebra with this primitive theory: after the publication of Proclus' *Commentary on the first book of Euclid's "Elements"* in 1533, this interpretation will be reaffirmed, because Proclus himself in the first Prologue reveals that the first ten theorems of Book II derive from Euclid's common

33 Common notions in Pacioli's version of Euclid include, among others: I. Things equal to the same thing are equal to each other. II. If we add the same things to equal things, they will still be equal. III. If we take equal things from equal things, the remainders will be equal. (*Euclidis Megarensis philosophi acutissimi*, f. 4v.)

notions and thereby constitute this primitive *mathesis*. In the sixteenth century, the theory of ratios and proportions, developed in the Middle Ages on the basis of Book V, was considered a natural extension of this primitive *mathesis*. Furthermore, for many authors, this way of looking at algebra as *mathesis* opens the way to its use in all mathematical sciences.

We can now go back to Pacioli's text and to his treatment of the second "case" of first degree equations:

> **1Ce is equal to 4Co**
> By consequence, if a *censo* is equal to 4 of its roots and its surface will be 4 of its roots, so its root will have four units.

The formulation is a little obscure and is a synthesis of a few aspects of Pacioli's thought. One of the difficulties is to translate his formulation into modern algebraic notation, that is, to replace the *cosa* by our symbol for the unknown "x" and the *census* by the unknown to the square "x^2." If we do it, we can see the previous rule as: "as many x will be equal to one x^2 as the units that will be contained in each of its x." To make sense of this statement, we have to see the roots as bidimensional.

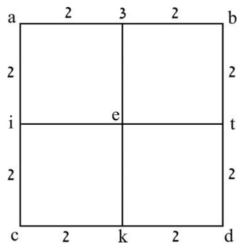
In the case of the equation in question, we need to apply the rule of solution:

5x will be equal to one x^2 and as many units will be contained in each one of its x.

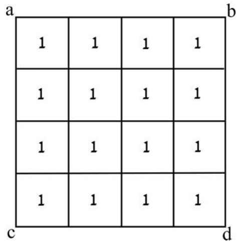
This formula seems a bit strange as a rule of solution because by itself it does not allow us to know how many units are there in the unknown. The situation is even more complex if we consider formulations 2 and 3 of Pacioli's rule, because it implies that in the equation "$5x = x^2$" not only the root or unknown is composed of five units, but there are five units. Here is a point of discontinuity between our notion of root and that expressed by Pacioli, because we now say that the root of 25 is 5, and not that 25 has 5 roots of 5 units.

But following Pacioli's text, we find another example clarifying further the notion of root.

One Ce is equal to 4 Cose
This example is accompanied by two diagrams which clarify Pacioli's notion of root: By consequence, if a Censo is equal to 4 of its roots and its surface will be 4 of its roots, so its root will have 4 of its units »…



As in the square ABCD which for each of its sides has four arms, so its surface is of 16 arms, which will become 4 of its roots. One of these is the square ae, the other 3t, the other ik, and the other kt.



Here, it is clear that Pacioli conceptualizes the square formed by 4 roots and not only that the root is formed by 4 units. This implies that the root is already understood as a surface.

In the case in question, 4 squares constitute a great square, the surface of which is 16 arms. Pacioli pursues his explanation by showing a second diagram that he presents as equivalent to the previous one:

> Hence the square ikcd is 16 and this is not absurd because 4 roots are equal to the area of this square.
>
> It follows that the root of this square is equal to the root of 4 of its roots
>
> That is the root of 16 is equal to the root of 4+4+4+4
>
> And because each root contains as many units as there are roots in the square, it is necessary to count 4 roots, that is 16 units; so that the root of the censo, becomes equal to the root of 16, that is 4, and this is the side of the square.

We can observe that there is at the same time a geometrical and an arithmetical intuition of Pacioli's idea of root. With respect to the previous example, this case is easier: we can show a square of surface 16 as formed by 4 squares of 4 units. This is what Pacioli says when he writes that "*the root of CE square is equal to four.*" In the case of a square of surface 16, the roots are already little squares belonging to a larger square as many times as the units which constitute the roots. This is not the case with the square of surface 25. It is not possible to show this square as constituted by 5 squares of 5 units each. When the coefficient of the unknown of the equality is a square number, then we can visualize the roots as squares constituting parts of the Censo: they will constitute parts of the Censo in the same amount as there are units in the root. In case such a coefficient is not a square number, we cannot visualize the roots as squares dividing the *censo* in as many times as there are units in the root. The root is still a geometrical object, but simply we cannot visualize it with a shape of a square being part of larger square constituted by the roots. Here, the second diagram proposed by Pacioli is crucial, because it allows us to visualize that each root is equivalent to a stripe, one side of which is the unit and the other is the amount of units of the root. Here, it appears clearly that the Root is a surface.

Pacioli expresses in this way a geometrical understanding of algebraic quantities or, more precisely, an understanding that pushes as far as possible both the arithmetical and the geometrical interpretation. This builds an arithmetical computation with geometrical numbers and is based only on Euclid's *Elements* Book II, but also interprets many theorems of Books V–IX.

This interpretation of algebra is based on continued fractions and geometric progressions: here, we have seen only the three first terms, with the unit. Michael Stifel's

Arithmetica Integra[34] developed this same interpretation: he used it to introduce the theory of powers of the unknown quantities; it was presented with the geometric progressions and their study with plane and solid numbers in the context of algebra. For Stifel, the purpose of his book is to display all the possible notions of number and potential computations and operations on numbers.

Also Pedro Nunes,[35] later in the century, gave a geometrical explanation of algebra which confirms and develops Pacioli's geometrical interpretation of algebraic quantities. Compared to Stifel's, Nunes' program, however, goes a step further: he wants to establish a new theory of ratios and proportions in algebraic terms. In order to do so, Nunes paid a similar attention to this introductive part of algebra: from the start, he gave the justifications and demonstrations of the solution formulas in an explicit imitation and transformation of Pacioli's text. Later, we shall compare his treatment of the second group of three equations to the one by Pacioli called composed, where the terms are constituted by three sorts of quantities: *Numero, Cosa* and Censo.

12.7 Pacioli: geometry and algebra

In order to understand how Pacioli used his theory of quantity providing the foundations of algebra, we must look at the proof of the solution of second degree equations.

Let's take another classical example or caso. This involves the actual use of common notions and continued proportions. It is quadratic:

> <u>**1Ce plus 10Co is equal to 39.**</u>
> Pacioli gives a geometrical demonstration of the equation of the first composed chapter. Article 10, *Summa*, f. 145v
> 1Ce plus 10Co is equal to 39, in our terms, $x^2 + 10x = 39$

Here, the idea is to represent the problem in terms of segments where the product is a square or a rectangle. Like in the previous case, we represent the Censo as a square without associating a number with it. Geometrically, we can visualize its side, which is the unknown magnitude, as root of the Censo, and build the rectangle which is the product of it and of 5 units, that is half of the number of Cose. We can in fact take two rectangles on two sides of the square. We can try to build a single figure with the three figures built so far (Ce + 5Co + 5Co). This is a classical figure attributed to the oldest Greek tradition, the Pythagoreans, that is a gnomon. We know that this figure has a numerical value of 39; therefore, we can think of it as constituted by 39 unitary

34 *Arithmetica integra*, Nurnberg, Petreius, 1544. Stifel was among the first and foremost followers of Martin Luther. Stifel's work had a crucial role in the new reformed education structured by Melanchthon, who prefaced Stifel's book.

35 Libro de Algebra en Arithmetica y Geometria, Anvers 1567. For more details on Nunes *doctrina* or arithmetico-geometrical justification, see my "Mathematical Progress or Mathematical Teaching? Bilingualism and Printing in European Renaissance Mathematics," in A. Bernard and C. Proust (eds.), *Scientific Sources and Teaching Contexts throughout history: Problems and Perspectives*. Dordrecht, Springer, 2014, pp. 187–214, as well as « Subtilissima arte. Pedro Nunes et l'algèbre de la Renaissance », colloque Pedro Nunes e a Ciencia do seu Tempo, 8–10 novembre 2002, Département de Mathématiques, Université de Coimbra, Coimbra.

squares. Starting with the gnomon, can we build another figure of which we know the numerical value? We can build another square of value 25.

Which figure constitutes the square of 25 with that of 39? What is its numerical value?

We can build a large square of numerical value 64. Starting with this, we can find the required solution.

Let us now go back to Pacioli's text.

12.8 Geometrical demonstration of the first composed chapter

First of all, Pacioli states that there is a substantial difference between the proofs of the rules of solution of the simple equation and of the composed equations.

> The first three simple chapters given here show by themselves the truth of each equality. And they have no need of another palpable demonstration if not by simple deduction ["deducimento"].
>
> But the other three composed chapters need a statement and a well thought demonstration, so that their truth appears in an adequate way.

This way of looking at the question is important and is the foundation for the solution of second degree equations. It involves two aspects: a justification in terms of common notions and measures attributed to geometrical figures and also a checking of the result.

Pacioli states that he wants to give "this demonstration for each of the 'chapters' and first for that of *li censi e cose se eguagliano al numero, como a dire*: 1 ce p 10 co son equali a 39." ("*The Censi and the Cose are equal to the number, that is to say: 1Ce plus 10 Co are equal to 39*"; in our terms, $x^2 + 10x = 39$.)

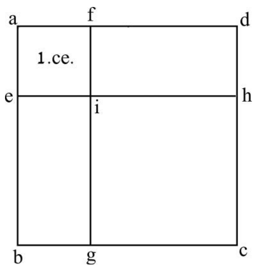
> We say that to find the value of censo and also of its cosa, having reduced the equation to 1 Ce, we divide by two the quantity of the cosa, here they are 10, so it is 5. We have to multiply this half by itself, which gives 25. To this we have to add the number found in the equation, that is 39. This will give 64. And take from this the root which is 8. And from this, we shall take the half of the cosa, that is 5, and it will be 3 for the value of the Cosa and this will be the root of the censo, that is 3, while the censo will be 9.
>
> (*Summa*, f. 145v–146)

This is the rule, the algorithm prescribing how to proceed with this type of equation. Separately, there will be the "case" and its demonstration. We can then see that the values found verify the equation. This corresponds to our procedure: we must first apply the procedure of solution which translated in our terms is $x = (\sqrt{b^2/4 + c}) - b/2$. This tells us to calculate the half of 10, then multiply it by itself, 25, which we must add to 39, it will make 64, of which we must take the root, that is 8. Finally, we subtract 5 from it and we get the "Thing" that is 3, while the Censo will be 9.

In order to prove this rule geometrically, Pacioli reasons in terms of calculation on segments to which he assigns a number. He takes a square in which by hypothesis the side is "more than 5 in quantity." Hence, we can choose on the side **ab** a point **e** so that **be** will be our segment 5 and **ea** the unknown quantity. Then, we proceed with the construction of the gnomon:

> For instance let the square abcd measuring for each side more than 5 in quantity. And on the side ab we have to mark the point e so that it will be 5 and the rest be the unknown quantity. We have to proceed in the same way on the side ad, by taking a point f at 5, so that on the side cd the point h and the point g on the further side bc […] ea and af are unknown. We must also connect the points e and h by a line traced parallel [*equidistantemente*] to the sides ad and bc. And similarly we must connect g and f by a straight line parallel to ab and cd.
>
> (*Summa* f. 146)

Pacioli gives the figure of the square abcd which we reproduce here from the page on which it appears in folio 146:



So far, Pacioli's proof does not need to quote the classical references about the fact that the segments of parallel lines are equal: the straight lines joining the points are in fact perpendicular to each other and parallel to the other sides. For Pacioli, the proof rests on a *notio communis:* **If we take the same thing to two equal things, the rests will still be equal, because of the common conception [*per la communa conceptione*].**

Pacioli continues:

> It follows that if we take the segments "5" from the equal segments da and ab, we shall have that ea and af are equal, though unknown. But the segments ae and fi will be also equal, because the straight line fg is equal to the straight line ab, from which we have taken respectively ig and eb. Similarly, eh is equal to ad and ih is 5, equal to fd by construction.

Hence, we arrive, **only by reference to a common notion**, to the conclusion that the initial square contains two little squares: ef, gh.

> Now let the censo of our unknown be the square ef, which is unknown to us because all its sides are unknown to us by hypothesis. Its root is each of its sides ei and if.

It is only by starting from this point that Pacioli refers to information contained in Euclid's *Elements* because he takes into consideration the rectangles built around the censo.

> According to the first definition[36] of book II of Euclid <*any rectangle parallelogram is contained by two segments which constitute a right angle*> each rectangle included in the first big square is contained by two segments one of which is Rx <root> and the other 5 co. Hence its area will be 5 co. The two rectangles are equal to each other.

Pacioli calls these rectangles "*suplementi*" and recalls proposition 43[37] of Book I of the des *Elements* as the foundation of this equality.

Thus, there will be three "surfaces," one of which is the Censo, what is asked for, the other two which are the supplements, 10 co. The case, or chapter, from Euclid which Pacioli is using to justifying the rule tells us that the sum of these three areas is equal to 39. If we add to these three areas the square *gh*, that is 25, built on 5, we get 64 for the large square *abcd*, "the root of which is 8 that is *ab*." If we subtract 5 from 8, we get 3 as root of the small square. The Censo will be 9. Pacioli verifies, by replacement, that the initial equation is valid for this solution, and states that "**this derives from the fourth [proposition] of Book II [of Euclid's *Elements*].**" Pacioli gives a precise reference to a specific definition in Book II, then to a theorem in Book I and finally to a theorem in Book II. In fact, if we compare Pacioli's proof to Fibonacci's proof in the *Liber Abaci*, we can have at first the impression that the two proofs are the same: also Fibonacci has this proof making use of "common conception," the expression used a few years later in Campanus' translation of Euclid, in turn, published only 12 years before Pacioli's *Summa*. Pacioli is actually quoting Fibonacci, using the same figure and the same letters; finally, both authors base their argument on the determination of the small square starting from the complements. But these similarities to Fibonacci's text make Pacioli's innovations evident. Pacioli's proof introduces explicit references to Euclidean definitions and propositions. Furthermore, while Pacioli gives a proper translation into vernacular Italian of Fibonacci's Latin text, he includes an explicit argument founded on the primitive theory with respect to both arithmetic and geometry, the practica speculativa which should be developed by algebra. In Fibonacci, this is present, but is implicit. Pacioli takes up the ancient argument of computation of segments but he elaborates it by referencing it: he makes clear what depends on theorems and what in common notions and Book II. In conclusion, we see Pacioli associating the basic algebraic operations of the equation, restoration and balancing, with the use of common notions and the ancient doctrine of the application of areas much more strongly than Fibonacci does. The application of areas, as we have mentioned, is incorporated in Euclid's *Elements* in Books II and VI, while the common notions appear in the short initial part of Book I of Euclid's *Elements* and they are precisely common to arithmetic and geometry. Pacioli, therefore, goes beyond Campanus' interpretation of Euclid associating arithmetic and geometry, arithmetical and geometrical ratios and proportions, and shows explicitly the Euclidean foundations of the rules of algebra. He is the first to conclude explicitly that it is possible to prove the rules of algebra by simple common notions and Book II

36 Omne parallelogrammum rectangulum, sub duabus rectum angulum comprehendentibus rectis lineis dicitur contineri. (Pacioli, *Euclidis megarensis philosophi acutissimi* Venice f. 9 v.)

37 Omnis parallelogrammi eorum quae circa dimetientem sunt parallelogrammorum supplementa, sibi invicem sunt aequalia. (f. 9: we find here the term "supplement," which in modern translations is replaced by "complement.")

and, given that these are independent from either arithmetic or geometry, algebraic theory is independent from theorems of arithmetic or geometry, rather is prior to both. As we have seen a few times, this depends on the composite tradition of the *Liber Mahameleth* on the one hand and on Campanus' version of Euclid's *Elements* on the other.[38]

By founding his argument on Campanus' interpretation of Euclid, Pacioli could assume that arithmetic and geometry have many theorems in common and this unified doctrine is the basis for human understanding of the world. Already Al Farabi had described Euclid as Pythagorean. This perspective complemented the ideas of Pythagorean thinkers rediscovered in Florence, such as Iamblichus. Later, Tartaglia will mention Pythagorean and Neo-Platonic mathematical texts edited by Lefèvre d'Etaples and transmitted by his teachings and his disciple Charles de Bovelles in the same direction. Luca Pacioli's *Summa* was reprinted in 1503. When Grynaeus's Basel edition of Euclid's *Elements* with Proclus' *Commentary*, printed in 1533, arrived in Paris, the context was definitely prepared to put it at the center of a new interpretation of algebra, like Pedro Nunes in Portugal. Before looking at later authors pursuing in Pacioli's direction, let us focus on other features of Pacioli's treatment. First, it is important to consider his idea of the powers of the unknown. As Pacioli puts it, we have to commit to memory the principle that **there are as many unities in a cosa as there are values of the cosa in a censo.**

12.9 The scale of powers and the meaning of algebra

Starting with what appears clearly as Pacioli's "second rule," that is, that *there are as many units in the Cosa as there are Cose in the Censo*, while working out the proof of the solution formulas, Nunes is able to give the sense of the equality of ratios between the powers of the unknowns; he got this following his treatment of simple equations with the Cosa and the Censo. The continued proportion introduced by Pacioli's second rule amounts to establish a numerical translation of the ratios between geometrical magnitudes. But it also gives the meaning of the equations in terms of the theory of ratios and proportions. This was very important for early modern mathematical practitioners in order to give a sense to their calculations: algebra could provide the numerical result, but in the ancient and early modern world, the only way to assure the mathematician that irrational[39] numbers had a meaning in the cosmos was to translate it into geometry.

Nunes is clear about the notion of number used in algebra:

> In this art is called number no matter which quantity, when we understand it as composed of units, be it an integer, a fractionary number, a Root or an irrational number.
>
> (p. 1r)

38 On Euclid's axioms and Euclidean traditions, see Vincenzo de Risi The Development of Euclid's Axiomatics. "The Systems of Principles and the Foundations of Mathematics in Editions of the Elements in the Early Modern Age." *Arch. Hist. Exact Sci.* 70 (2016): 591–676.

39 About irrationals among sixteenth-century algebraist, see Roy Wagner, "The Natures of Number in and Around Bombelli's Algebra." *Arch. Hist. Exact. Sci.* 64 (2010): 485–523.

This is explained more generally by Nunes at folio 9r, with reference to proposition 16 of Book IX of Euclid's *Elements*. It is with Pedro Nunes that we come to a consistent presentation of the theory, with demonstrations founded on various books of Euclid. As Nunes puts it in a letter to the reader, at the end of the book:

> I give demonstration of all the rules that I use, and I don't make reference to any other author than Euclid, and in the proper passages I find more than enough.

At folio 9r, Nunes writes:

> **Demonstration of the second rule of composed equations.**
> This figure and demonstration was used by Euclid in the fourth proposition of Book II to prove that, if the AC line were divided into 2 parts, as in point B, the square of the entire AC line would be equal to the squares of both parts in conjunction with the rectangle that the parts contain twice. And Campanus demonstrated the same on proposition 16 of Book 9 in number of indivisible units which will serve our purpose demonstrated in this way.

Notice that Nunes has the same nomenclature as Pacioli for the powers of the unknown as for many other aspects of algebra. This is not surprising also because Nunes is said to have written his algebra a few decades before the actual publication in 1567. The list of these quantities, taken further than the ones appearing in the equations of the second degree, is a geometric progression:

Powers of unknowns

		Denomination
Cosa	2	***Unit***
Censo	4	**2**
Cubo	8	**3**
Censo di censo	16	**4**
Relato primo	32	**5**
Censo di cubo	64	**6**

Co.2.Ce.4.Cu.8.Ce.ce.16.Re.p.32.Ce.cu. o Cu.ce.64. etc.
1. 2. 3. 4. 5. 6.

Already Pacioli in his *Summa* had used this same progression (*Summa*, f. 143 v, see our figure 2). This is first based on f. 37, when he shows how to find the sum of geometric progressions and takes as an example the progressions of powers of 2. Later in the *Summa*, he deals with continued proportions (f. 85). In his version of Euclid's *Elements* based on Campanus, Pacioli also gives the definition of plane (numbers contained by two segments) and solid numbers in Book VIII; these he calls continuous quantities and the theorems of Book IX about the construction of geometric progressions with continuous quantities. Hence, Pacioli provides the basic elements of this matter, establishing an explicit and intrinsic connection between the theorems in Book II, about segments, and the theorems in Book IX, about plane and solid numbers. In the first relevant passage, Pacioli is talking about dividing a square number into two numbers in order to get remarkable products. In this **numerical** context, he writes that this rule depends on Book II and says that

> If you wanted to know the cause and the origin of the rule mentioned above, you should know that they are given in Book II. In which he says that **each time that a quantity** is divided into two parts.

Here, we see already at this early stage that Pacioli is talking about numbers, but supposedly the Euclidean reference is talking about **quantity**, a generic quantity. The idea is to found algebra on this ancient geometric progression of the powers of the unknown which, as a continuous proportion, includes in fact not only numbers but generic quantities. Later in the *Summa*, in the treatment of algebra, Pacioli introduces plane and solid or figurate numbers at the level of the justification of the solution of equations, as we have seen. Stifel will pick up and develop extensively also this aspect in his book on algebra. Again a full interpretation of this idea and a reason for it appears clearly in the context of Nunes' **doctrina** (this is the way he calls his arithmetico-geometrical foundation of algebra and the explanation of the procedures). Nunes should be acknowledged as the mathematician who actually spelled out the connection of algebraic powers of the unknown to the plane and solid numbers and continuous proportions in Euclid and included the justifications of the solution rules for second degree equations. As we mentioned earlier, Campanus gave an interpretation of Euclid's *Elements* which associated the arithmetical and the geometrical books of Euclid's *Elements*, a reading of Euclid which became common through the late Middle Ages and Renaissance. Starting with Pacioli, this is the case for other sixteenth-century authors such as Stifel and, in the French algebraic tradition, Jean Borrel in 1559 (*Logistica*) and Petrus Ramus (Pierre de la Ramée, 1515–1572). In his *Algebra* (1560), he had insisted that algebra was an aspect of Euclid's *Elements*. In fact, Ramus provides other historical evidence of this view which is not mentioned by Klein and yet comes from a mathematician who definitely belonged to the same mathematical milieu as Gosselin and Viète, that is, Petrus Ramus: in the *Scholae mathematicae* printed in Basel in 1569,[40] he developed an argument against the logical order of Euclid's *Elements*. According to him, the great dialectician, expert of *Posterior Analytics*, the *Elements* do not respect the order of learning (that is, the order of *mathesis*). In particular, Euclid made the mistake of not starting from the logical beginning, but with the treatment of triangles in Book I. He should have started instead with Book II, which is logically prior because it deals with a general doctrine which could then be specified with arithmetic or geometry. Ramus stated this after an extensive exegesis of Proclus' *Commentary of Book I of Euclid Elements*, where Proclus mentions the theorems belonging to the original doctrine. Ramus has already published his own *Algebra* in which he talked about figurate numbers as the powers of the unknown. He can develop the section on Book II in the *Scholae Mathematicae* by saying that this book elaborates on figurate numbers. He concludes that the arithmetic of figurate numbers *is not simply arithmetic, but* ***geometric arithmetic*** (*Scholae Mathematicae* p. 194).

40 Petrus Ramus, *Scholae mathematicae*. Basel, Eusebius Episcopus, 1569.

Maybe Nunes had also access to Tartaglia's *Euclide Megarense*, the Italian version of the *Elements* in which he had expressed a similar view about Euclid, even though without the criticism on the order.[41]

Nunes himself said that his manuscript changed Pacioli's order, because he wanted to give a justification of the solution formulas at the beginning. This allowed Nunes to attribute algebra to Euclid, as did Jean Borrel in his *Logistica* in 1559. Notice also that the theoretical point is very similar to the one expressed by Gundisalvi in *De unitate et uno*, because we connect arithmetic and geometry through the division in unit squares. Furthermore, Nunes begins his discourse on algebra with a reference to Aristotle's distinction between continuous and discrete quantities. Besides the **doctrina** and the justifications for the solutions of equations, what structures Nunes' whole book is the theory of proportions. Proportions have the central place in it.

In fact, this foundation of the progression was very explicitly taken up ten years after Nunes by Guillaume Gosselin at the beginning of his *De Arte Magna* (1577). Gosselin writes in his third chapter "What is algebra," after the definition of the subject:

> The whole sense of it is in proportion, given that all this divine art emanates from that beautiful eighth theorem of the ninth book of Euclid, and depends on it, and it gives a demonstration of the multiplication of the names of this science, as we shall show in our version of Tartaglia's Arithmetic.
>
> (*De Arte Magna* f. 3)

Jacob Klein noticed this sentence by Gosselin, and wrote that "*surprisingly, [Gosselin] sees the foundation of this science in Euclid IX, 8*" (Klein, p. 263). Gosselin mentions this theorem also in his version of Tartaglia of 1578, in the definition of Root.[42] In fact, theorem IX, 8 in Tartaglia's version[43] says that

> If there are many numbers in continued proportion, the third from the unit will be a square, and from it on skipping one number, and the fourth from the unit will be a cube, and from it on always skipping two numbers, and also the seventh from the unit is squarecube and so on always skipping five numbers will be always a squarecube.

In other words, the powers of a number are in a geometrical progression. Any number. The starting number is a unit, but the first root can be a general number: integer, rational or irrational. This concerns the algebraists because a consequence of this is that the powers of the unknown are in a geometrical progression. So, Gosselin stressing the importance of the theorem does not appear so surprising anymore. We see it present in Pacioli's mind because of his rule: there are as many

41 Notice that now there is no copy of Tartaglia's translation of Euclid's *Elements* at the library in Coimbra. However, it is possible that Nunes had one, or also that he used Etienne's version, extant at the library.

42 Guillaume Gosselin, *L'Arithmetique de Nicolas Tartaglia Brescian*. Paris, Gilles Beys, 1578, Seconde partie.

43 Niccolò Tartaglia, *Euclide Megarense Philosopho*. f. 123v.

units in the unknown as there are unknowns in the square power, that is, 1: x = x:x^2; in other words, the relations among powers are established by a continued proportion. And this is also the reason behind the famous sentence by Viète stressed by Jacob Klein: *An equation is a solution of a proportion and every proportion is the construction of an equation* (GMTOA p. 160). Instead of *construction*, we could also say *constitution*, the literal translation of Viète's *constitutio*. I think that in this sentence of the *Isagoge*, Viète intended to stress the difference between the "stuff" we are talking about, that is, proportions between quantities, and the computation we make on them in order to solve a problem and find a value, that is, the equation.[44]

Nunes also mentions proposition 16 of Book IX among the Euclidean theorems which give a foundation for algebra: if between two numbers prime to each other there will be as many numbers as we wish in a continuous proportion, there will be as many numbers between these and the unit. As it appears in Campanus' version of Euclid's *Elements*, this proposition and its corollaries *idem proponit*, that is, propose the same things as the first theorems of Book II. A consequence of this proposition is what we would call the development of the power of the binomial, that is, again Book II, proposition 4 and the justification of the solution for second degree equations. In other words, Nunes reaffirms strongly the foundation of algebra on the connection between the theory of proportions and the primitive discipline of the application of areas.

Notice that the geometric progression so important for sixteenth-century algebraist is not the same as the one used by the Diophantus, his editor Xylander and Viète, with additive exponents, but it has multiplicative exponents instead. While it is a matter of notation, it has the consequence that in the Diophantine nomenclature we know the rank of each power, and only that one Instead, precisely because of the propositions in Book IX, in this older nomenclature, we have the information that the squares and the cubes recur regularly and there is a way to know the position in the progression and not only the features of the number (or power) by its name. What counted for early modern mathematicians though is that these are a particular kind of number, that is, plane and solid, defined as such in the definition of Book VIII, in Campanus' version of Euclid's *Elements*. Numbers, plane and solids are defined after the theory of ratios established in Book V. Here, we have the definitions of numbers as roots of squares, while in Book X numbers can be irrational. In this way, first Pacioli, then Tartaglia, Nunes and Gosselin have articulated their idea of figurate number as power of a general root. The number is general (a unit, an integer, a rational, an irrational number), and a term as a definite amount of that general number, conceived, in turn, as definite amount of units. How many units? The answer is Pacioli's second rule; there are as many units in the Cosa as there are Cose in the Censo. Any equation is a continued proportion. This is enough to proceed with the mathematics. This treatment by Nunes and by his follower Gosselin seems to clarify fully the foundations that Pacioli intended to provide for algebra. But it also gives a possible content to Klein's statement about Viète's species.

44 Notice however that Viète uses here *aequalitas*, which mostly meant the result of an equation.

12.10 Viète's species

Klein writes:

"Vieta's law of homogeneity is concerned rather with the fundamental fact that every 'calculation' since it does, after all, ultimately depend on 'counting off' the basic units, presupposes a field of *homogeneous monads*" (Klein pp. 173–174). The case of Diophantus' *Arithmetic* is different; his equations contain *arithmoi* and monads. Klein says that for the *logistice speciosa*, the fundamental presupposition of counting off the basic units needs to be especially stressed. This is the reason why Viète emphasizes the role of the law of homogeneity as the foundation of the "analytical art," in contrast to the ancient analysts who used to have different kinds in the same equation. Here, Klein has a sentence which marks a turn in his essay:

> Thus it appears that the concept of the species is for Viete, its universality notwithstanding, irrevocably dependent on the concept of '*arithmos*'. He preserves the character of the *arithmos* as a "number of…" in a peculiarly transformed manner. While every *arithmos* intends immediately the things or the units themselves whose number it happens to be, his letter sign intends directly the general character of being a number which belongs to every possible number.
>
> (Klein p. 174)

I would make a point in terms of sixteenth-century mathematicians: Viète's species preserves the character of "being a number of" for every root of any plane or solid number. Or, in our terms, any root must be a number, be it an integer, or a fraction, or a root, or an irrational root. It is understood as composed by units and we can perform the operations on it. In this context, Viète's law of homogeneity is a reminder that the quantities are of different dimensions: each term is a plane or solid number which should be multiplied or divided in order to have the same dimension as the other terms. But this same law tended to become obsolete in what will be the new context: Descartes started his *Géométrie* with a reduction to a unit but his analytic geometry had more similarities than differences with Fermat's analytic geometry (*Ad Locos Planos et Solidos Isagoge*), written in pure Vìetian language.

We have seen different contexts in which algebra, or rather the quantities presented in equations, or the powers of the unknowns, are justified by Euclidean geometry and arithmetic. I have also stressed how this justification through Euclid was, in turn, based on a theory of quantities prior to arithmetic and geometry but common to both. This first discipline was identified by sixteenth-century people with geometric algebra and Jordanus' arithmetical identities, according to Campanus' interpretation of Books II and VII. We have also seen that the "scale" or continued proportion of the powers of the unknowns allows the mathematicians to visualize it through the application of areas: Roots as sides of Squares, Squares as Roots of Cubes and so on. This continued proportion established an explicit connection between propositions in Book II, about calculation on segments, and plane numbers and solid numbers. Book IX provides the application of the theory of ratios and proportions to plane and solid numbers, and here we come to a theorem famous among sixteenth-century algebraists, about geometric progression defining the reciprocal connections between powers of the unknown.

We have said and given examples of the fact that sixteenth-century algebraists had a theory, the *practica speculativa*, a geometric algebra consisting first of *communes conceptiones*, that is, Euclid's common notions, to which were added theorems from Book II on geometric algebra and from Book IX on continued proportion. Pacioli called it *practica speculativa*, Tartaglia *pratica speculativa*, Nunes *doctrina*. It included the first ten theorems of Book II, a few definitions of Book VII and the propositions in Book IX that reflected Book II. The "entities" were the powers of the unknowns represented as segments, squares, cubes and parallelepipeds. Michael Stifel (1487–1567) illustrates this theory very well with his diagrams and geometric progressions (see Appendix 3). This mathematical structure corresponded to a philosophical frame providing a foundation for algebra. The mathematical structure was Euclidean, based on Campanus, whereas the ontology corresponding to it can be traced back to Gundisalvi, from *De Unitate* to *On the Sciences* and the *Liber Mahamelet*. I consider these new elements as new evidence, in fact a confirmation of Klein's theses. There is just one point in which my interpretation is in contrast with what seems to be Klein's view: that Viète's stipulations define the new mathematical objects, the species, providing an axiomatic theory of them. Axioms in Greek mathematics are the common notions. In fact, they are the first few items in Viète's list of stipulations. Only postulates and not common notions can, in Euclid's *Elements*, establish something analogous to an axiomatic theory concerning geometrical objects, that is, to define them by their operations. But this is not the point here. Species, or Viète's general quantities as we call it, could not be simply defined by axioms. The existence of the objects functioning according to what the axioms prescribe is not in question for Viète, because all the axioms are taken from Euclid and the main, non-trivial example was geometry. In fact, the idea that axioms make mathematical objects, that the objects (in particular, functions) exist through definition, is a typical late nineteenth-century idea. Klein seems to suggest that Viète started this process; being a classics specialist, he knew that Viète the humanist could only admit mathematical objects already existing in the ancient ontology. I think that Klein described here a shift in Viète: Viète used the tools of his predecessors but by collecting all the axioms, the same objects entered into a new configuration. The axioms here are an extension of the common notions we have seen in action in Pacioli's algebra. Also, the quantities were general and for an astronomer they should be as close to geometry as possible. But the novelty is that irrationals are admitted: the main notion is the structure of the problem, that is, the proportion expressed by an equation, so numbers are just roots of the equation; then, the purpose is that any kind of root must be admitted.

12.11 Conclusion: Kleinian theses on Viète

What is new with respect to Klein's theses on Viète?

The novelty in the historiography of algebra in the last decades is a new emphasis on the role of Arabic thinking in connection with algebra, from mathematics to logic and ontology. Present already in the mathematical texts such as the commentaries on Euclid and on Al Kwarizmi's treatises, this body of knowledge was transmitted through the translations taking place mostly in the Iberic peninsula. The process of transmission also concerned philosophical and logical texts, again Greek and Arabic, but also a new cultural context, with Jewish and Christian intellectuals interpreting Arabic science and shaping it according to new priorities. The Arabic texts are sometimes autonomous (al Farabi) and sometimes presented as commentaries (Ibn Rushd). In this sense, one could imagine a new book by Klein, in fact a second volume of GMTOA in which instead of the gap between the Greeks, Diophantus and modern algebraists, we could see the slow process of absorption, elaboration and transmission of Greek culture to Arabic scientists and from Arabic to Latin and Hebrew scientists, culminating in the Italian Renaissance circles of late fifteenth century and beginning sixteenth. This process from Toledo to Venice is much more intercultural than it was depicted by earlier scholarship. First of all, Medieval Arabic scientists are now acknowledged as original thinkers not only in algebra but in many mathematical sciences, having an important role in Western philosophy and theology.[45] Furthermore, the Hebrew transmission and original contribution also has emerged forcefully in the historical picture, and we know now the importance of mathematics in Jewish texts: treatises, translations, commentaries, textbooks. We also see that Christians and Hebrew thinkers collaborated in the production of translations and commentaries. Here I tried to convey how this complex intercultural process taking place among a variety of religions, languages and disciplines had a strong impact on the meaning of mathematics, and in particular on the epistemological shift Klein described in GMTOA. The technical innovation at the origin of the shift can be situated in the introduction of the notion of number as root, allowing calculations with irrationals, in the constitution of a theory of equations. But the philosophical innovations leading to species as general quantities are in the blend of cultures transmitting the classics and yet transforming them, and creating a new culture of imitation, translation and commentaries. Philological studies and techniques did contribute to the construction of Renaissance algebraic quantities, but not as French algebraist described, as a *restitutio*, a reconstruction of a Greek or primitive art of the Golden Age: that world being lost, what emerges is the linguistic power of conception, the space created by the new dialectic. The *practica speculativa* involves the computation on segments as the primitive discipline prior to arithmetic and geometry and foundation for algebra: this is an assumption on human knowledge coming from dialectic, as the theory of first and second intentions. All of these points are identified as crucial by Klein, though unknowingly that they originated in the Arabic language. This is also the case for the very notion of continuous number, common to Gundisalvi, Pacioli, Tartaglia, Stevin and Viète. We might notice in particular that this posits Tartaglia's thought as a crucial moment in the process from Viète to Wallis, as a Venitian agent of transmission.

Klein's theses in fact would benefit from the hypothetical second volume because it is in this process that *prima et secunda intentio* were invented. Klein's theses would also benefit from the fact that geometric algebra as a primitive science was an Arabic idea.

In 1996, I had stressed the anti-Arabism of the French Humanists as opposed to the continuist view of the Italian mathematicians, more connected with the Mediterranean cultures through the widespread abacus schools. These acknowledged the Arabic origins of algebra, while French algebraists looked for a Greek origin and believed to be ready to reconstruct it. This Mediterranean view is even more emphasized by the present study. Arabic Hebrew and Christian understanding of algebra

45 It is important to mention the works by Alain de Libéra and Marwan Rashed in this connection.

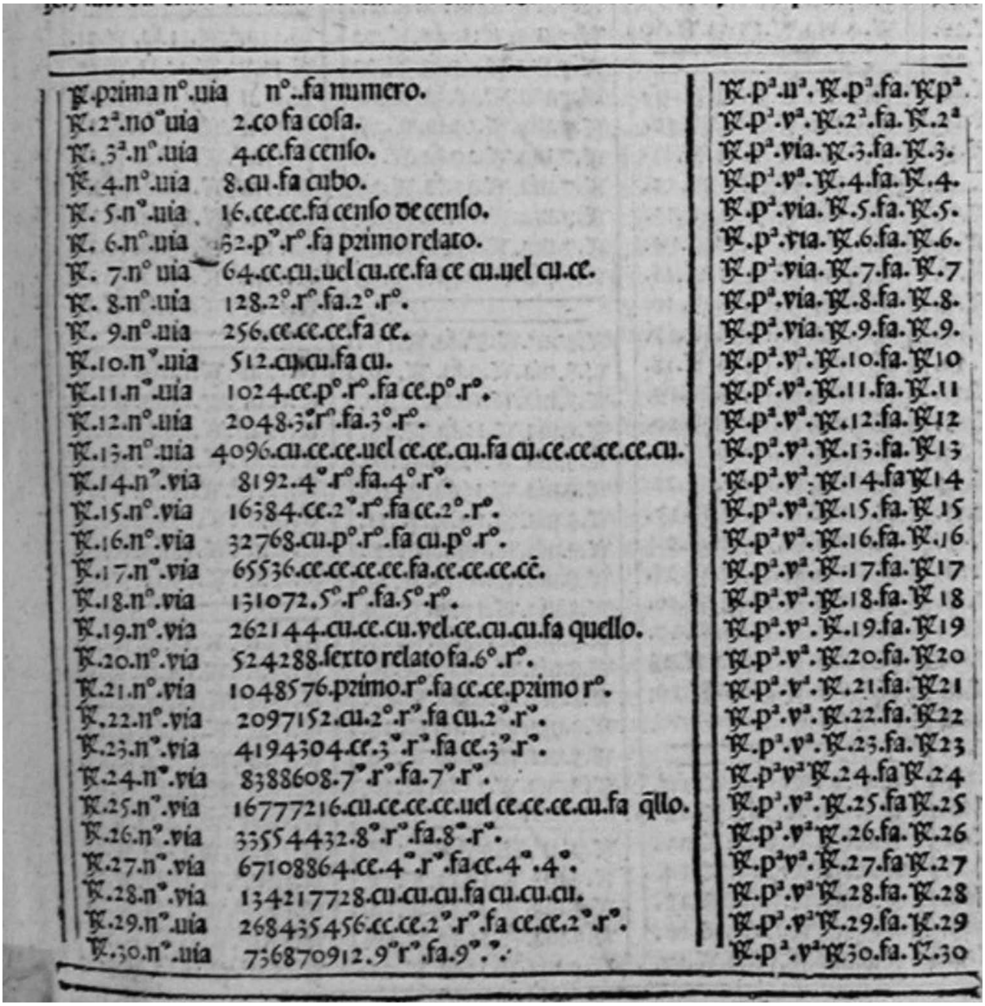


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Figure 12.1 Luca Pacioli Summa de Arithmetica geometria, proportioni et proportionalità. f. 143.

based on Al Farabi's classification of sciences and Ibn Rushd's reading of Aristotle appear in the major books of the two Euclid's scholars Pacioli and Tartaglia and also in Nunes. To found algebra on the *Elements* meant to ascribe it to that tradition (Figure 12.1).

Acknowledgments

I wish to thank my students, Jean-Marie Coquard, Bertrand Paoloni and David Silva Labra, always very active and full of ideas in my EHESS seminar, and Burt Hopkins, who accepted my invitation to the EHESS, coorganized our workshops in Paris and invited me to contribute to this journal.

Appendix 1

From what we have seen so far, we can make a **conjecture** relevant to algebra, that is, that philosophical thinking about unit, number, quantity and *prima et secunda* intention went as follows:

	1st Intention	*2nd Intention*
Number of the Ancients	Unit + Collection of units	Idea of unit + Idea of number as collection of units
Diophantus	Sign of the Idea of Number in general	
Al Kwarizmi	*Sign of each Number (Arabic Numerals). Thing without notation)*	*Idea of general quantity (Thing)*
Number of the Arabs, Fibonacci, Campanus	*Unit generating the series, numerate number*	*Res, Census, Cubus (General) quantity*
Al Banna, Al Qalasadi	*Sign of the Thing (Cosa) as (General) quantity*	
Pacioli, Tartaglia, Nunes, Gosselin	*Sign of the general quantity, Sign of several unknowns*	*Species of general quantity*
Stevin, Viète, Descartes, *Fermat*	Sign of the species of the general quantity	Species of the general quantity *(classes of equations)*

Appendix 2

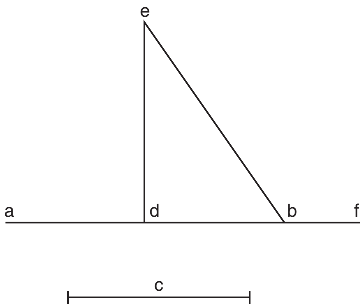
Nunes' demonstration "new and perfect" of the solution for a second degree equation

The demonstrations of the last three rules are very clear. However, they could be criticized by some opponent saying that in the demonstration of the first one, we presuppose that a censo with the cosas in any quantity that they are can be equal to any number, understanding number as we have defined at the beginning of this book, and that this presupposition is not true. Therefore, it will be necessary to prove it.

Let the line AB be the number of the Cosas, and the number that was put equal to the cosas with the Censo that has for square root (Squared Side) the line C. We will divide the line AB in half at point D and that from that same point the line DE is projected, in a way that it generates right angles with the line AB, and that it is equal to the line C.

From point E, we will draw to point B, the straight line EB, thus constructing right triangle EDB.

Extend the DB line as much as necessary to cut DF equal to the EB line.



And we say that the line BF is of side a Censo that together with the Cosas whose number is AB is equal to the proposed number that has for "Squared Side" the line C. And this will be the demonstration:

The square of BE is equal to the square of ED and the square of DB, both together, due to proposition 47 of Euclid's first book.

And since BE and DF are equal, the square of DF will be the same size as the squares of BD and DE.

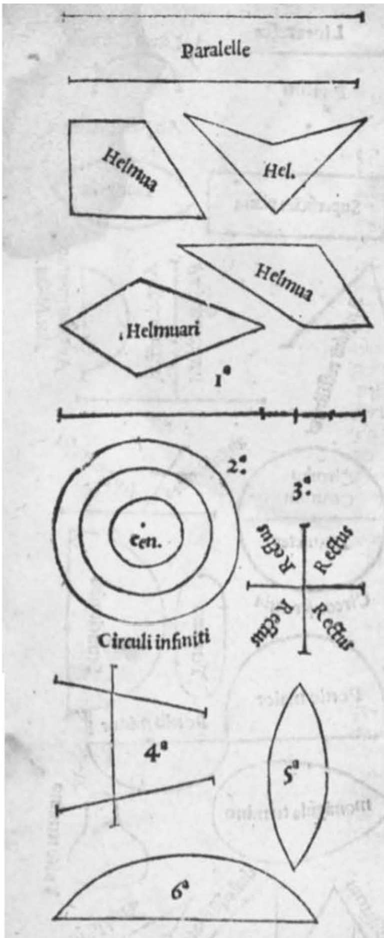
That same square of DF is equivalent to the squares of BD and BF plus the double of the rectangle comprised by DB and BF **by the 4 propositions of the second book**. Therefore, we will therefore remove from those two sums, which by common notion are equal, the common square from the DB line, leaving the square of the DE line equal to the sum of the square BF with the double of the rectangle comprised by DB and BF.

And because DB is half the number of Cosas, we put BF lado of the censo and it will be therefore the rectangle understood by DB and BF half the value of the Cosas, and the double of this rectangle will be the whole value of them.

And the Censo with the Cosas will be equal to the square of the line DE that we put equal to the line C whose square we put that was the number, that in principle we had put as equal to the Cosas in conjunction with the Censo, and this is what we wanted to prove.

A.12.1 Common conceptions of the mind

1 Two things equal to the same thing are equal to each other.
2 If equal things are added to equal things, the sums are equal.
3 If equal things are subtracted from equal things, the remainders are equal.
4 If you subtract equal things from unequal things, the remainders will be unequal.
5 If you add equal things to unequal things, the sum will be unequal.
6 If two things are equal to the same thing, they will also be equal to each other.
7 If two things are both half of the same thing, they are equal to each other.
8 If we apply one thing to another and it does not exceed it, the two things will be equal.
9 Any whole is greater than its part.



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equilaterum atq; rectangulum. Alia est tetragonus longus. q̃ est figura rectangula. sed eqlatera non est. Alia est helmuaym. que est equilatera. sed rectangula non est. 21. ¶ Alia est similis helmuaym que opposita latera habet equalia. atq; oppositos angulos equales. idem tamen nec rectis angulis nec equis lateribus cõtinetur. Preter has autem omnes quadrilatere figure helmuariphe nominãtur. 22. ¶ Equidistantes linee sunt q̃ in eadem superficie collocate atq; in alterutram partem protracte non conueniunt. etiam si in infinitum protrahãtur.

Castigator.

¶ A diffinitione superficiei superius posita excipitur circularis que postea specialiter inferius ponitur que non continetur terminis: & ideo eius termini non sunt linee sed linea & etiam monangula excipitur &c q̃uis multe curuilinee / sperice connexe concaue & plane lineis contineantur quibus posset diffinitio illa aplicari vt patet intuenti.

Petitiones sunt quinq;. 1. ¶ A quolibet puncto in quemlibet punctum rectam lineam ducere atq; lineam definitam in cõtinuum rectumq; quantũlibet protrahere. 2. ¶ Super centrum quodlibet quantumlibet occupando spacium circulum designare. 3. ¶ Omnes rectos angulos sibiinuicem esse equales. 4. ¶ Si linea recta super duas lieas rectas ceciderit duoq; anguli ex vna parte duobus rectis angulis minores fuerint istas duas lineas in eandem ptem protractas proculdubio coniunctum iri. 5. ¶ Duas lineas rectas supficiem nullam cõcludere.

Communes animi cõceptiões sunt hee. 1. ¶ Que vni & eidem sunt equalia & sibiinuicem sunt equalia. 2. ¶ Et si eq̃libus equalia addant' tota quoq; fient equalia. 3. ¶ Et si ab equalibus equalia auferantur que relinquuntur erunt equalia. 4. ¶ Et si ab inequalibus equalia demas que relinquuntur erunt inequalia. 5. ¶ Et si inequalibus equalia addas ipsa quoq; fient inequalia. 6. ¶ Si fuerint due res vni equales ipse sibiinuicem erunt equales. 7. ¶ Si fuerint due res quarum vtraq; vnius eiusdem fuerit dimidium vtraq; erit equalis alteri. 8. ¶ Si aliqua res alicui supponatur applicetur q; ei nec excedat altera alteram. ille sibiinuicem erunt equales. 9. ¶ Omne totum est maius sua parte.

Castigator.

¶ Ista que vni & eidem &c. quam Euclides hic acomodat quantitatibus infra in. v. qnti libri acomodat proportionibus vt ibi. ⁊ Prima & sexta differunt tanq̃ inclusum & exclusum vel sicut abstractum & concretum vel sicut mensura & mensuratum vel sicut prima & secunda intentio siue res & species rei.

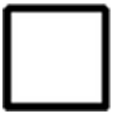
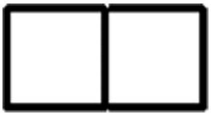
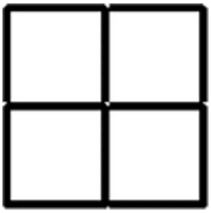
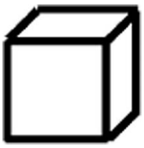
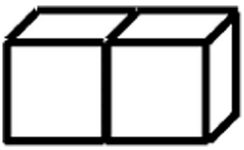
Verba Campani.

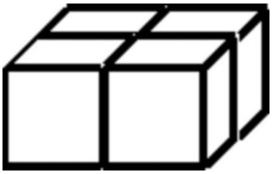
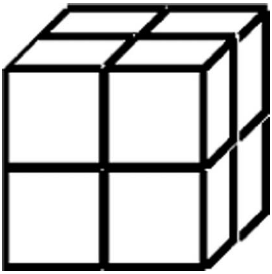
Sciendum est autem q; preter has animi conceptiones siue communes scientias multas alias que numero sunt incomprehẽsibiles pretermisit Euclides: quarum hec est vna. ¶ Si due quantitates equales ad quamlibet tertiam eiusdem generis cõparent' simul erunt ambe illa tertia aut eq̃ maiores aut eq̃ minores aut simul eq̃les. ¶ Ite alia Q uãta est aliq̃ q̃titas ad quãlibet aliã eiusdẽ generis tantã esse quãlibet tertiã ad

Campanus
Preclarissimus liber elementorum Euclidis perspicacissimi : in artem Geometrie incipit quam foelicissime. Erhard Ratdolt. Venice 1482.

Appendix 3

Stifel's ladder

Unitas		1		
Radix		2	Ra	X
Census		4	Ce	X^2
Cubus		8	Cu	X^3
Censicensus		16	Cece	X^4

Primum Relatum		32	Re	X^5
Censicubicus		64	Cecu	X^6